Patches and left inverses

Let $X \subset \mathbb{R}^n$ and $x \in X$. We recall that we defined a patch on X around x to be the data of an open set $U \ni x$, and open set $P \subseteq \mathbb{R}^d$ and a bijection $f : P \to X \cap U$ such that f is a C^1 immersion and f^{-1} is continuous.

This note will prove that, given this data, we can shrink U and P to U' and P' which still have the above properties, so that there is a C^1 map $g : U' \to P'$ with $g \circ f = \mathrm{Id}$.

We proved a weaker version of this on September 25: Simply using that f is an immersion, we can shrink U to U' and P to P' such that $f(P') \subset U'$ and there is a C^1 map $g : U' \to P'$ with $g \circ f = \mathrm{Id}$. This equation forces $f : P' \to f(P')$ to be bijective. However, we need not have $f(P') = X \cap U'$ without the hypothesis that f^{-1} is continuous.

Now, assuming that f^{-1} is continuous, we explain how to fix this. Let P', U' and g be as in the previous problem. Since f^{-1} is continuous, we know that $(f^{-1})^{-1}(P') = f(P')$ is open in X, for the subspace topology. By definition, this means that there is some U'' open in $\mathbb{R}^n$ such that $f(P') = X \cap U''$. We replace U'' by $U'' \cap U'$ so that we may assume $U'' \subseteq U'$. So we now have $f : P' \to U''$, $g : U'' \to P'$ obeying $g \circ f = \mathrm{Id}$ and $f(P') = X \cap U''$.

It remains to check that $f : P' \to X \cap U''$ is a bijection. Since $g \circ f = \mathrm{Id}$, we know f is injective. But also we forced $f(P') = X \cap U''$, so f i surjective to $X \cap U''$. $\square$