## Identities involving the Lie Bracket

Recall that, for two $n \times n$ matrices $X$ and $Y$, we put $[X, Y] = XY - YX$.

**Problem 29.** We define $\mathrm{ad}_X$ to be the linear map $Y \mapsto [X, Y]$. So, if we were going to write this as a matrix, it would be an $n^2 \times n^2$ matrix. Show that

$$e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{\mathrm{ad}_X^n(Y)}{n!}.$$

We recall the formula

$$(D\exp)_X(Y) = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{X^j Y X^{n-1-j}}{n!}.$$

**Problem 30.** Let $X$ be the diagonal matrix with entries $x_1, x_2, \ldots, x_n$. Let $i \neq j$ and let $Y$ be the matrix with 1 in position $(i, j)$ and 0 everywhere else. Show that

$$(D\exp)_X(Y) = \sum_{n=0}^{\infty} \frac{\mathrm{ad}_X^n(Y)}{(n+1)!}\, e^X = e^X \sum_{n=0}^{\infty} \frac{(-1)^n \mathrm{ad}_X^n(Y)}{(n+1)!}.$$