

MAPS BETWEEN LIE GROUPS

Let $G \subset \mathrm{GL}_m(\mathbb{R})$ and $H \subset \mathrm{GL}_n(\mathbb{R})$ be Lie groups. Let \mathfrak{g} and \mathfrak{h} be their Lie algebras. Let $\phi : G \rightarrow H$ be both

- (1) a group homomorphism, meaning $\phi(\mathrm{Id}_m) = \mathrm{Id}_n$ and $\phi(g_1 g_2) = \phi(g_1) \phi(g_2)$.
- (2) a smooth map, meaning there is an open set $U \supseteq G$ and a C^∞ function $\tilde{\phi} : U \rightarrow H$ such that $\tilde{\phi}$ restricts to ϕ on G .

We write $D\phi$ for the map $\mathfrak{g} \rightarrow \mathfrak{h}$ given by $D\tilde{\phi}$ restricted to \mathfrak{g} .

Problem 33. For $X \in \mathfrak{g}$, show that $\exp((D\phi)X) = \phi(\exp X)$.

Problem 34. For X and Y in \mathfrak{g} , show that $(D\phi)\big([X, Y]\big) = \big[(D\phi)(X), (D\phi)(Y)\big]$.