

THE LIE BRACKET

Problem 27. Let $X \subset \mathbb{R}^n$ be a manifold and $x \in X$. Suppose that $\gamma : (-\delta, \delta) \rightarrow X$ is a smooth path with $\gamma(0) = x$, $\gamma'(0) = 0$ and $\gamma''(0) = \vec{v}$. Show that $\vec{v} \in T_x X$. Check that this does not hold without the hypothesis $\gamma'(0) = 0$.

Let $G \subset \mathrm{GL}_n(\mathbb{R})$ be a Lie group, meaning G is simultaneously a subgroup of G and a manifold in $\mathbb{R}^{n^2}$. Let $X, Y \in \mathfrak{g}$ and define $[X, Y] = XY - YX$.

Problem 28. Show that $[X, Y] \in \mathfrak{g}$. Hint: Consider $e^{tX}e^{tY}e^{-tX}e^{-tY}$.