Application: The spectral theorem!

The goal of this worksheet is to prove the following: Let H be an $n \times n$ symmetric matrix. Then there is $U \in O(n)$ and a diagonal matrix D such that

$$H = UDU^{-1}.$$

Choose constants $r_1 < r_2 < \cdots < r_n$ and, for an $n \times n$ matrix Z, define

$$f(Z) = \sum_{i=1}^{n} r_i Z_{ii}.$$

Define $g : GL_n \to \mathbb{R}$ by $g(U) = f\left(U^{-1} H U\right)$.

Problem 19. Show that there must be some $U_0 \in O(n)$ where g is minimized.

Problem 20. Let $U \in O(n)$. Show that Dg_U restricts to 0 on $(TO(n))_U$ if and only if $U^{-1}HU$ is diagonal. (Hint: I recommend parametrizing $O(n)$ near U by $J \mapsto U \exp(J)$ for $J \in \mathfrak{so}(n)$.)

Problem 21. Explain why you have proved the spectral theorem: If H is a symmetric $n \times n$ matrix then there is $U \in O(n)$ and a diagonal matrix D with $H = UDU^{-1}$.

If you have more time:

Problem 22. Prove the singular value decomposition theorem: For any $H \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, there are U and $V \in O(n)$ and a diagonal matrix D with $H = UDV^{-1}$.