Correction to Monday

Let $X \subset \mathbb{R}^n$ and let d be a nonnegative integer. Last time, we gave a too weak version of Condition (1). Here is the correct one:

For every $x \in X$, there exists

- $U \ni x$, an open subset of $\mathbb{R}^n$
- P, an open subset of $\mathbb{R}^d$
- f, a bijection between P and $U \cap X$

such that

- $f : P \to U$ is a C^r-immersion and
- $f^{-1} : U \cap X \to P$ is continuous.

See Chapter 5, Section 23 in Munkres.

We want to show that Condition (1) implies condition (4), namely: For every $x \in X$, there exists:

- V and W open subsets of $\mathbb{R}^d$ and $\mathbb{R}^{n-d}$, with $x \in V \times W$
- q a C^r function $V \to W$

such that $(V \times W) \cap X = \{(v, q(v)) : v \in V\}$.

Proof: Let $p = f^{-1}(x)$. After rearranging rows, we may assume that $(Df)_p$ is of the form

$$(Df)_p = \begin{bmatrix} A \\ B \end{bmatrix}$$

with A an invertible $d \times d$ matrix. Let π_1 and π_2 be the projections of $\mathbb{R}^n$ onto the first d-coordinates and the last $n-d$ coordinates. Let $s = \pi_1 \circ f$. So $(Ds)_x = A$. By the inverse function theorem, we can shrink P to P' such that $s : P' \to s(P')$ is bijective with C^r inverse. Put $\tilde{V} = s(P')$. Define $q : \tilde{V} \to \mathbb{R}^{n-d}$ as $\pi_2 \circ f \circ s^{-1} = \pi_2 \circ f \circ (\pi_1 \circ f)^{-1}$.

Now, for $v \in \tilde{V}$, by definition, we have $v = s(y) = \pi_1(f(y))$ for some $y \in P'$. Then $q(v) = \pi_2(f(y))$. So $(v, q(v)) = (\pi_1(f(y)), \pi_2(f(y))) = f(y)$. So f takes $\tilde{V}$ into $X \cap U$. But we need to work harder to make sure every point of $X \cap U$ is of the form $(v, q(v))$.

Since f^{-1} is continuous, we can shrink U to U' such that $f(P') = X \cap U'$. We can then find an open set W such that $\pi_1(x) \times W \subset U'$ and we can shrink $\tilde{V}$ to V such that $q(V) \subset W$. So $q : V \to W$ and, for each $v \in V$, we have $(v, q(v)) \in (V \times W) \cap X$. Conversely, let $(v, w) \in (V \times W) \cap X$. So $(v, w) = f(y)$ for $y \in P'$. So $s(y) = \pi_1(f(y)) = v$ and $s^{-1}(v) = y$. So $q(v) = \pi_2(f(s^{-1}(v)) = \pi_2(f(y)) = w$, as desired.