Problem Set 0 – due September 15

This problem set is meant to use entirely ideas from Math 295-296, so that you can begin thinking about it immediately. All answers must be justified (computations shown, or proofs given, as appropriate).

Many of my problem sets, including this one, will contain a problem which I imagine most students will not solve. Struggling with challenging problems, and sometimes failing, is how you strengthen your mathematical ability. Difficult problems will often reappear later with hints.

I am glad to answer questions and offer help; please feel free to e-mail me and to come to office hours or make an appointment at some other time.

See the course website for policy on collaboration.

0. What is something you found really beautiful in your last math class? What is something that you found really confusing?

1. If we have a container of N molecules of a di-atomic gas (like oxygen), the pressure, temperature and volume of the container are related by $PV = NkT$, where k is a constant known as Boltzmann's constant. The entropy of this gas is
$$S = Nk \log\left(\frac{VT^{5/2}}{CN}\right)$$
where C is a physical constant. The amount of gas stays fixed throughout this question.
   (a) If we hold the volume of the container fixed, what is $\partial S/\partial P$?
   (b) If we hold the temperature of the container fixed, what is $\partial S/\partial P$?
   (c) Suggest an improvement to the notation $\partial S/\partial P$. Then realize that the notation has been used since 1786, so you are probably stuck with it.

2. For which of the following functions does the limit as $(x, y) \to (0, 0)$ exist? (Prove your answers correct.)
$$f(x,y) = \frac{xy}{x^2+y^2} \quad g(x,y) = \frac{x^3}{x^2+y^2} \quad h(x,y) = \frac{x^2y}{x^4+y^2}.$$

3. (This question should be a review of material you have seen before.) Let V be a vector space over the field $\mathbb{R}$ (real numbers). Define V^* to be the set of linear maps $V \to \mathbb{R}$. For f and g in V^*, define the element $f + g$ in V^* by $(f + g)(v) = f(v) + g(v)$. For f in V^* and a scalar $a \in \mathbb{R}$, define the element af in V^* by $(af)(v) = af(v)$. Check that these definitions make V^* into a vector space.

4. Let V be an n-dimensional real vector space with basis $e_1, e_2, \ldots, e_n$. For $\vec{v} = \sum a_i e_i$ in V, we define
$$|\vec{v}|_1 = \sum |a_i| \qquad |\vec{v}|_2 = \sqrt{\sum |a_i|^2} \qquad |\vec{v}|_\infty = \max(|a_i|).$$
   (a) Show that there are constants c_1, C_1, c_2 and C_2 such that
$$c_1|\vec{v}|_\infty \le |\vec{v}|_1 \le C_1|\vec{v}|_\infty \qquad c_2|\vec{v}|_\infty \le |\vec{v}|_2 \le C_2|\vec{v}|_\infty.$$
   (b) Let $f_1, \ldots, f_n$ be a second basis for V. For $\vec{v} = \sum b_j f_j$, set $|\vec{v}|^f_\infty = \max(|b_j|)$. Show that there are constants d and D such that
$$d|\vec{v}|_\infty \le |\vec{v}|^f_\infty \le D|\vec{v}|_\infty.$$

5. Let f be a function $\mathbb{R} \to \mathbb{R}^2$; we write $f(t) = (f_1(t), f_2(t))$. Show that $\lim_{h \to 0} \frac{f(t_0+h)-f(t_0)}{h}$ exists if and only if f_1 and f_2 are differentiable at t_0.

6. For a function f from $\mathbb{R}$ to $m \times n$ matrices, we define $f'(t)$ in the usual manner as $\lim_{h \to 0}(f(t+h) - f(t))/h$.
   (a) Let $A(t)$ and $B(t)$ be differentiable functions from $\mathbb{R}$ to $n \times n$ matrices. Give a formula for $\frac{d}{dt}A(t)B(t)$ in terms of $A(t)$, $A'(t)$, $B(t)$ and $B'(t)$.
   (b) Let $A(t)$ be a differentiable function from $\mathbb{R}$ to $n \times n$ matrices, so that $A(t)$ is invertible for all t. Give a formula for $\frac{d}{dt}A(t)^{-1}$ in terms of $A(t)$ and $A'(t)$.

7. We identify $\mathbb{R}^9$ with the set of 3×3 matrices. Which of the following subsets of $\mathbb{R}^9$ are open? Which are closed?
   (a) The set of invertible matrices.
   (b) The set of orthogonal matrices.
   (c) The set of matrices with rank 1.

8. For A be a $k \times k$ matrix, define $|A| = \max_{1 \le i,j \le k} |A_{ij}|$.
   (a) Show that $|AB| \le k|A||B|$.
   (b) Show that, for any $k \times k$ matrix A, the sum $\sum_{n=0}^{\infty} \frac{A^n}{n!}$ is absolutely convergent.

9. Let $f(x)$ be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$ with $f(0) = 0$. Define

$$g(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases}.$$

   Prove that $g(x)$ is continuous.

10. Let $f(x)$ and $g(x)$ be continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that, for $x \neq 0$, the derivative $f'(x)$ exists and $f'(x) = g(x)$. Prove that $f'(0)$ exists and equals $g(0)$.