PROBLEM SET 1 – DUE SEPTEMBER 22

See the course website for policy on collaboration.

1. Let $x(t)$ be a differentiable real valued function of t, obeying $x(t)^5 - x(t) = t$. What are the possible values for $x'(0)$?

2. Let $g(x,y) = \frac{x^3 y}{x^6+y^2}$ for $(x,y) \neq (0,0)$ and $g(0,0) = 0$.
   (a) Show that, for any a and $b \in \mathbb{R}$, the function $g(at, bt)$ is differentiable, and $\left.\frac{dg(at,bt)}{dt}\right|_{t=0} = 0$.
   (b) Show that $g(x,y)$ is not differentiable.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix A. Recall that the ***rank*** of f is defined to be the dimension of the subspace $f(\mathbb{R}^n)$ of $\mathbb{R}^m$. The goal of this problem is to prove that $\mathrm{rank}(f) \geq k$ if and only if there is a $k \times k$ submatrix of A with nonzero determinant.

   We introduce some helpful notations: For I any subset of $\{1, 2, \ldots, m\}$ and J any subset of $\{1, 2, \ldots, n\}$, let A_{IJ} denote the submatrix $(A_{ij})_{(i,j) \in I \times J}$. We'll write f_{IJ} for the linear map $\mathbb{R}^{|J|} \to \mathbb{R}^{|I|}$ corresponding to A_{IJ}.
   (a) Let $|I| = |J| = k$. Show that $\mathrm{rank}(f_{IJ}) = k$ if and only if $\det A_{IJ} \neq 0$.
   (b) Show that, for any I and J, we have $\mathrm{rank}(f) \geq \mathrm{rank}(f_{IJ})$.
   (c) Let $r = \mathrm{rank}(f)$. Show that there is $J \subset \{1, 2, \ldots, n\}$ with $|J| = r$ and $\mathrm{rank}(f_{\{1,2,\ldots,m\}J}) = r$.
   (d) Let $r = \mathrm{rank}(f)$. Show that there is $I \subset \{1, 2, \ldots, m\}$ and $J \subset \{1, 2, \ldots, n\}$, both of size r, so that $\mathrm{rank}(f_{IJ}) = r$.
   (e) Show that $\mathrm{rank}(f) \geq k$ if and only if there is a $k \times k$ submatrix of A with nonzero determinant.

4. Let A be a linear map from $\mathbb{R}^m \to \mathbb{R}^n$.
   (a) Suppose that $m \leq n$ and $\mathrm{rank}(A) = m$. Show that we can find a linear map $B : \mathbb{R}^{n-m} \to \mathbb{R}^n$ such that the map $A \oplus B : \mathbb{R}^m \oplus \mathbb{R}^{n-m} \to \mathbb{R}^n$ is invertible. (Here $(A \oplus B)(\vec{v} + \vec{w}) = A(\vec{v}) + B(\vec{w})$ for $\vec{v} \in \mathbb{R}^m$ and $\vec{w} \in \mathbb{R}^{n-m}$.)
   (b) Suppose that $m \geq n$ and $\mathrm{rank}(A) = n$. Show that we can find a linear map $B : \mathbb{R}^m \to \mathbb{R}^{m-n}$ such that the map $A \oplus B : \mathbb{R}^m \to \mathbb{R}^{m-n} \oplus \mathbb{R}^n$ is invertible. (Here $(A \oplus B)(\vec{v}) = A(\vec{v}) + B(\vec{v})$. Note that $A(\vec{v}) \in \mathbb{R}^m$ and $B(\vec{V}) \in \mathbb{R}^{m-n}$.)

5. Let V and W be finite dimensional real vector spaces, let A be an open subset of V and let $f : A \to W$ be a continuous function. The function f is called ***proper*** if, for any $R > 0$, the set $\{x \in A : |f(x)| \leq R\}$ is compact. (Recall that a subset of V is compact if it is closed *in* V and bounded.)
   (a) Show that the image of a proper map is closed.
   (b) Let $g = g_n z^n + g_{n-1} z^{n-1} + \cdots + g_0$ be a polynomial with complex coefficients and g_n nonzero. Let $f(x,y) = (\mathrm{Re}(g(x+iy)), \mathrm{Im}(g(x+iy)))$. Show that f is proper.
   (c) The map f is called ***open*** if, for any open $U \subset A$, the image $f(U)$ is open. Show that, if f is proper and open, then $f(A) = W$.

   If you suspect this is heading for a proof of the fundamental theorem of algebra, you are right. More next time.

Turn over for one more great question!

6. This question proves that the matrix exponential is differentiable, with

$$(D\exp)_X(Y) = \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{X^m Y X^{n-m-1}}{n!}.$$

I review definitions and results which I expect we will see in class on Friday the 15th: For a real matrix X, we define $|X| = \sqrt{\sum_{i,j} X_{ij}^2}$. I expect we will show, and you may use, that $|XY| \leq |X| \cdot |Y|$. I expect we will show, and you may use, that $|X+Y| \leq |X| + |Y|$.

(a) Show that there exists a constant C such that $|\exp(Y) - \mathrm{Id} - Y| \leq C|Y|^2$ for all Y with $|Y| < 1$.

(b) Deduce that $(D\exp)_0(Y) = Y$.

(c) Let X be a $k \times k$ nonzero real matrix with $|X| = R > 0$ and let Y be a $k \times k$ real matrix with $|Y| = r < R$. Show that

$$\left| (X+Y)^n - X^n - \sum_{m=0}^{n-1} X^m Y X^{n-m-1} \right| \leq 2^n R^{n-2} r^2.$$

(d) Fix $0 < r < R$. Show that there is a constant C (dependent on r and R) such that, if $|X| \in (r, R)$ and $|Y| < r$, we have

$$\left| \exp(X+Y) - \exp(X) - \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{X^m Y X^{n-m-1}}{n!} \right| \leq C|Y|^2.$$

(e) Deduce that

$$(D\exp)_X(Y) = \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{X^m Y X^{n-m-1}}{n!}.$$