PROBLEM SET 10 – DUE DECEMBER 8

See course website for policy on collaboration. This is the last problem set!

1. Let γ be a circle of radius r in $\mathbb{R}^2$ centered at 0, oriented counter clockwise. Compute
   (a) $\int_\gamma x dx$
   (b) $\int_\gamma x dy$
   (c) $\int_\gamma \frac{y dx - x dy}{x^2+y^2}$

2. (a) Let $f(x,y)$ be a C^∞ function on $\mathbb{R}^2$. Show that
$$\frac{\partial}{\partial y}\int_{x=a}^{b} f(x,y)dx = \int_{x=a}^{b} \frac{\partial f}{\partial y}(x,y)dx.$$
   (b) Let g and h be C^∞ functions $\mathbb{R}^2 \to \mathbb{R}$, obeying $\frac{\partial g}{\partial y} = \frac{\partial h}{\partial x}$. Define
$$F(x,y) = \int_{s=0}^{x} g(s,0)ds + \int_{t=0}^{y} h(x,t)dt.$$
   Show that
$$\frac{\partial F}{\partial x} = g \text{ and } \frac{\partial F}{\partial y} = h.$$
   (c) On $\mathbb{R}^2 \setminus \{(0,0)\}$, put
$$g(x,y) = \frac{y}{x^2+y^2} \text{ and } h(x,y) = \frac{-x}{x^2+y^2}.$$
   Check that $\frac{\partial g}{\partial y} = \frac{\partial h}{\partial x}$. Show that, nonetheless, there does not exist $F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ such that $\frac{\partial F}{\partial x} = g$ and $\frac{\partial F}{\partial y} = h$. (One approach is to use your computation from 1(c).)

3. Let V be a finite dimensional real vector space, $A \subset V$ an open subset and α be a differentiable vector field on A, meaning a C^1 function $A \to V$. For $x \in A$, we define the divergence $(\nabla \cdot \alpha)(x)$ to be the real number $\mathrm{Tr}(D\alpha)_x$. So, in coordinates, $\nabla \cdot \alpha = \sum_i \frac{\partial \alpha_i}{\partial x_i}$. This problem checks basic properties of divergence; the parts are not very related to each other.
   (a) ~~Let W be another vector space of the same dimension as V, B an open subset of W and $\phi : A \to B$ a diffeomorphism. Let α be a vector field on A. Define a vector field β on B by $\beta(y) = (D\phi)_{\phi^{-1}(y)}\alpha(\phi^{-1}(y))$. Check that $(\nabla \cdot \alpha)(x) = (\nabla \cdot \beta)(\phi(x))$.~~ Deleted for being false! If you'd like a challenge, give a counter-example. One dimension will do it!
   (b) Let $V = \mathbb{R}^n$ and let α be a vector field on $\mathbb{R}^n$. Let C be the cube $[0,1]^n$. Let $L_i = \{(x_1,\ldots,x_n) \in C : x_i = 0\}$ and $R_i = \{(x_1,\ldots,x_n) \in C : x_i = 1\}$. Check that
$$\int_C (\nabla \cdot \alpha) = \sum_i \int_{R_i} e_i \cdot \alpha - \sum_i \int_{L_i} e_i \cdot \alpha$$
   where e_i is the standard basis of $\mathbb{R}^n$ and $\cdot$ is the ordinary dot product.

4. In this problem, we will prove the identity that, for X an $n \times n$ matrix,

$$\det \exp(X) = e^{\operatorname{Tr} X}.$$

Let SL_n be the set of $n \times n$ matrices with determinant 1 and let ${}_n$ be the set of $n \times n$ matrices with trace 0.

(a) Show that SL_n is an $(n^2 - 1)$-fold and ${}_n$ is its Lie algebra.

(b) Show that, if X is an $n \times n$ matrix with $\operatorname{Tr} X = 0$, then $\det \exp(X) = 1$. (This is just a matter of remembering which IBL result to cite.)

(c) Show that, for any scalar c, we have $\det \exp(c \, \mathrm{Id}_n) = e^{\operatorname{Tr}(c \, \mathrm{Id}_n)}$.

(d) Show that, for any matrix X, we have $\det \exp(X) = e^{\operatorname{Tr} X}$.

5. Let

$$\begin{aligned} \mathcal{Q} &= \left\{ (\alpha, \beta) : \begin{array}{rcl} 0 &<& \alpha, \beta \\ 2\alpha + \beta,\ \alpha + 2\beta &<& \pi \end{array} \right\} \\ \mathcal{A} &= \left\{ (x, y) : \begin{array}{rcl} 0 &<& x, y \\ e^{-x} + e^{-y} &>& 1 \end{array} \right\} \end{aligned}$$

Consider the system of equations

$$\begin{aligned} e^{-x} \sin \alpha &= e^{-y} \sin \beta \\ e^{-x} \cos \alpha + e^{-y} \cos \beta &= 1 \end{aligned} \qquad (*)$$

(a) Show that there is a bijection $(x, y) \to (\alpha(x, y), \beta(x, y))$ from $\mathcal{A}$ to $\mathcal{Q}$ so that $(x, y, \alpha(x, y), \beta(x, y))$ obeys the equations $(*)$. (This is mostly the main result of Problem Set 2, Problem 3, just explain how to modify it for this problem.)

(b) Show that $\mathcal{Q}$ and $\mathcal{A}$ have the same area.

(c) Show that the area of $\mathcal{A}$ is

$$\int_0^\infty -\log(1 - e^{-x}) dx$$

and show that this integral equals $\sum_{n=1}^\infty \frac{1}{n^2}$. (Yes, you need to justify moving the sum past the integral.)

(d) Compute the area of $\mathcal{Q}$.