Problem Set 2 – due September 29

See the course website for policy on collaboration.

1. Let $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^3 + 3z^4 = 6,\ x^2 + y^2 + z^2 = 3\}$.

   Show that there is an interval $(1-a, 1+a)$ around 1 and smooth functions y and $z : (1-a, 1+a) \to \mathbb{R}$ so that $y(1) = z(1) = 1$ and $(x, y(x), z(x))$ is in C for all $x \in (1-a, 1+a)$.

2. Let $f(x, y) = y^3 - x^5 - x^3$. Is there a differentiable function $g : \mathbb{R} \to \mathbb{R}$ so that $g(0) = 0$ and $f(x, g(x)) = 0$?

3. Suppose that we have a differentiable function $(x, y) \mapsto (\alpha(x, y), \beta(x, y))$ from an open set in $\mathbb{R}^2$ to $\mathbb{R}^2$ so that

$$e^x \cos\alpha(x, y) + e^y \cos\beta(x, y) = 1 \qquad e^x \sin\alpha(x, y) + e^y \sin\beta(x, y) = 0.$$

   (a) Compute the matrix of partials $\begin{pmatrix} \partial\alpha/\partial x & \partial\alpha/\partial y \\ \partial\beta/\partial x & \partial\beta/\partial y \end{pmatrix}$.

   (b) Show that, if $e^{x_0} + e^{y_0} > 1$, $e^{x_0} + 1 > e^{y_0}$ and $e^{y_0} + 1 > e^{x_0}$, then there is an open set U containing (x_0, y_0) for which such functions α and β exist.

4. Let $A \subset \mathbb{R}^n$ be an open set and $f : A \to \mathbb{R}$ a smooth function. Let $a \in A$.

   (a) Suppose that $f(a) = \min_{x \in A} f(x)$. Show that $(Df)_a = 0$.

   Let $x_1, \ldots, x_n$ be the coordinates on $\mathbb{R}^n$ and set $f_{ij} = \frac{\partial^2 f}{(\partial x_i)(\partial x_j)}(a)$.

   (b) Suppose again that $f(a) = \min_{x \in A} f(x)$. Show that, for all $(v_1, \ldots, v_n) \in \mathbb{R}^n$ we have $\sum_{i,j} f_{ij} v_i v_j \geq 0$.

   (c) Suppose that $(Df)_a = 0$ and, for all $(v_1, \ldots, v_n) \in \mathbb{R}^n \setminus \{(0, 0, \ldots, 0)\}$ we have $\sum_{i,j} f_{ij} v_i v_j > 0$. Show that there is an open set $A' \ni a$ such that $f(a) = \min_{x \in A'} f(x)$.

5. This question is about considering polynomials in $\mathbb{C}[z]$ as differentiable maps $\mathbb{R}^2 \to \mathbb{R}^2$. For a function $g : \mathbb{C} \to \mathbb{C}$, we will write $\tilde{g}$ for the corresponding function $\tilde{g}(x, y) = (\text{Re } g(z + iy), \text{Im } g(x + iy))$, mapping $\mathbb{R}^2$ to $\mathbb{R}^2$. For $\alpha \in \mathbb{C}$, write μ_α for the linear map $z \mapsto \alpha z$ from $\mathbb{C} \to \mathbb{C}$. So $\tilde{\mu}_\alpha$ is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$.

   (a) Show that $\tilde{\mu}_\alpha$ is invertible for $\alpha \neq 0$.

   (b) For $g_n, g_{n-1}, \ldots, g_0 \in \mathbb{C}$, let $g(z) = \sum_{j=0}^n g_j z^j$ and define $g'(z) = \sum_{j=0}^n j g_j z^{j-1}$ (the formal derivative of g). Show that $(D\tilde{g})_{(x,y)} = \tilde{\mu}_{g'(x+iy)}$.

   (c) Let U be open in $\mathbb{C}$ and let $g(z)$ be a nonzero polynomial in $\mathbb{C}[z]$. Suppose that $g'(z)$ is nonzero on U. Show that $g(U)$ is open in $\mathbb{C}$.

   We will now work to prove that $g(U)$ is open without the hypothesis that $g'(z) \neq 0$.

   (d) Suppose that $g(0) = 0$ but g is not identically 0. Let k be the minimal index for which $g_k \neq 0$. Show that there is an open set $V \ni 0$ on which we can write $g(z) = r(z)^k$ with $r : V \to \mathbb{C}$ smooth and $(D\tilde{r})_0$ invertible.

   (e) In the above notation, show that there is an $\epsilon > 0$ such that $g(V)$ contains the disc of radius ϵ around 0.

   (f) Let g be any nonzero polynomial in $\mathbb{C}[z]$. Show that, if U is open, then $g(U)$ is open.

6. This question addresses the following question: Let X and Y be topological spaces and let $f : X \times Y \to \mathbb{R}$ be a continuous function. Set $g(x) = \inf_{y \in Y} f(x, y)$. Is g continuous?

   (a) Define $f(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $f(x, y) = \exp(-x^2 y^2)$. Define $g(x) = \inf_{y \in Y} f(x, y)$. Compute g and show that it is NOT continuous.

   (b) Suppose that Y is compact (if it would make you more comfortable, you may also assume X and Y are metric spaces). Show then that g is continuous.

   (c) In (a) there was not actually a point where the infinimum $\inf_{y \in Y} f(x, y)$ was achieved (for $x \neq 0$). If we do not require that Y is compact, but do require that, for all x there is a y_x such that $f(x, y_x) \leq f(x, y)$ for all $y \in Y$, will g necessarily be continuous?