Problem Set 3 – due October 6

See the course website for policy on collaboration.

1. We begin with a practical application of the Implicit Function Theorem! The Global Positioning System involves roughly 20 satellites which transmit regular signals down to earth. These signals travel at the speed of light, c. So, if a GPS receiver sitting at (x, y, z) receives a signal at time t which was sent from a satellite at (x_1, y_1, z_1) at time t_1, then

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = c^2(t - t_1)^2.$$

The satellites have highly accurate clocks and time stamp their signals and their orbits are known with high accuracy. So the receiver can have confidence in knowing when and where a signal was sent from. However, its own clock may have drifted by some unknown amount b. So, if the receiver measures a signal received at time s, the actual time might be $s + b$.

Our receiver gets signals from four satellites, which it measures as arriving at times s_1, s_2, s_3 and s_4. The signal which arrives at s_i was sent from a satellite at (x_i, y_i, z_i) at time t_i. The receiver wants to compute (x, y, z, b) as functions of (s_1, s_2, s_3, s_4). The receiver also knows roughly where and when it is, so it may assume (x, y, z, b) lie in a small open set in $\mathbb{R}^4$.

   (a) Under what conditions will (x, y, z, b) be locally given by a smooth function of (s_1, s_2, s_3, s_4)? You may leave your answer as a determinant without expanding it.

   (b) What will $\partial x/\partial s_i$, $\partial y/\partial s_i$, $\partial z/\partial s_i$ and $\partial b/\partial s_i$ be? You may express your answer in terms of matrix operations without expanding them.

2. Show that an open subset of $\mathbb{R}^n$ is an n-dimensional manifold. (Hint: This is very short.)

3. Verify that the following are manifolds and give their dimensions:

   (a) The set of $(x_1, y_1, x_2, y_2) \subset \mathbb{R}^4$ such that $x_1^2 + y_1^2 = 1$ and (x_2, y_2) is on the line which is tangent to the unit circle at (x_1, y_1).

   (b) The set of $n \times n$ matrices with rank $n - 1$.

4. Let S be the sphere $\{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Let $\mathbb{RP}^2$ be the image of S under the map

$$f(x, y, z) = (x^2, y^2, z^2, xy, xz, yz).$$

   (a) Is the map $f : S \to \mathbb{RP}^2$ injective? If not, when do we have $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$?

   (b) Show that $\mathbb{RP}^2$ is a 2-dimensional manifold in $\mathbb{R}^6$.

5. Let $X \subset \mathbb{R}^n$ be a d-fold and let $x \in X$. Show that there is an open set $U \ni x$ such that $X \cap U$ is connected.

6. ~~Let V be a finite dimensional vector space and $X \subset V$ a d-fold in V. Define~~

$$\cancel{TX = \{(x, \vec{v}) \in V \oplus V : x \in X,\ \vec{v} \in T_x X\}.}$$

~~Show that TX is a $2d$-fold in $V \oplus V$.~~

7. Let r, m and n be positive integers, with $r \leq m, n$. We'll be looking at $m \times n$ matrices divided into blocks as shown below:

$$X = \begin{array}{cccc} \overbrace{}^{r \text{ columns}} & \overbrace{}^{n-r \text{ columns}} & \\ A & B & \}r \text{ rows} \\ C & D & \}m-r \text{ rows} \end{array}$$

(a) Given A, B, C with A invertible, there is precisely one D such that X has rank r.

(b) Show that the D in the previous problem is a smooth function of A, B and C.

(c) Show that the set of rank r matrices is a $mr + nr - r^2$ dimensional submanifold of the set of $m \times n$ matrices.