Problem Set 4 – due October 13

See the course website for policy on collaboration.

1. (a) Compute the tangent plane to the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$ at $(1, 1, 1)$.
   (b) Compute the tangent line to the curve in $\mathbb{R}^3$ parametrized by $t \mapsto (t, \sin t, \cos t)$ at $(0, 0, 1)$.

2. The aim of this question is to describe a Möbius strip and prove it has one side. Let $c : \mathbb{R}_{>0} \times \mathbb{R}^2 \to \mathbb{R}^3$ be the cylindrical coordinates map:
$$c(r, h, \theta) = (r \cos \theta, r \sin \theta, h).$$
Let $\tau : \mathbb{R} \times (-1, 1) \to \mathbb{R}_{>0} \times \mathbb{R}^2$ be given by
$$\tau(t, u) = (2 + u \cos(t/2), u \sin(t/2), t).$$
Define $M = (c \circ \tau)(\mathbb{R} \times (-1, 1))$.
   (a) For any $t_0 \in \mathbb{R}$, show that $M = (c \circ \tau)([t_0, t_0 + 2\pi] \times (-1, 1))$. Describe which points of $[t_0, t_0 + 2\pi] \times (-1, 1))$ are mapped to the same point of M. (This part explains why we are calling M a Möbius strip.)
   (b) Show that $c \circ \tau$ is an immersion.
   (c) Show that M is a 2-fold.
   (d) For $t \in \mathbb{R}$, compute a basis $u(t)$, $v(t)$ for $T_{(c \circ \tau)(t,0)} M$.
   (e) Let $w(t) : \mathbb{R} \to \mathbb{R}^3$ be a continuous function such that $|w(t)| = 1$ and $w(t)$ is perpendicular to $T_{(c \circ \tau)(t,0)} M$ for all t. Show that $w(t) = -w(t + 2\pi)$.

   In other words, if we start with a unit normal vector pointing out from one side of M, and go around M, it comes back pointing the other way.

3. Let X and Y be subsets of $\mathbb{R}^n$ which are a d-fold and an e-fold respectively. Suppose that, at every point z of $X \cap Y$, the intersection of linear spaces $T_z X \cap T_z Y$ has dimension $d + e - n$. Show that $X \cap Y$ is a $(d + e - n)$-fold.

4. The first steps of this problem concern a skew-symmetric matrix $M = \begin{bmatrix} 0 & x & -y \\ -x & 0 & z \\ y & -z & 0 \end{bmatrix}$. We put $\theta = \sqrt{x^2 + y^2 + z^2}$.
   (a) Show that
$$\exp(M) = \mathrm{Id} + \sin \theta \tfrac{M}{\theta} + (1 - \cos \theta) \tfrac{M^2}{\theta^2}.$$
   (Hint: Notice a relationship between M and M^3.)
   (b) What axis does $\exp(M)$ rotate around and by what angle? (Prove your answer is correct.)
   We now consider exp as a function on $\mathfrak{so}(3)$, with coordinates as above:
   (c) Show that exp is injective on the open ball $\{x^2 + y^2 + z^2 < \pi^2\}$. Describe which pairs of points in the closed ball $\{x^2 + y^2 + z^2 \leq \pi^2\}$ are sent to the same rotation under exp.
   (d) Show that, if U is any open set containing $\begin{bmatrix} 0 & 2\pi & 0 \\ -2\pi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then exp is **not** injective on U.
   (e) Let $U = \{2\pi - \frac{\pi}{10} < x < 2\pi + \frac{\pi}{10},\ -\frac{\pi}{10} < y, z < \frac{\pi}{10}\}$. Show that $\exp(U)$ is **not** open in $SO(3)$.