Problem Set 5 – due October 27

See course website for policy on collaboration.

1. (a) Set

$$g(x,y) = \begin{cases} 1 & x < y < x+1 \\ -1 & x-1 < y < x \ . \\ 0 & \text{otherwise} \end{cases}$$

Show that

$$\int_{x=0}^{\infty} \left(\int_{y=0}^{\infty} g(x,y) dy \right) dx \neq \int_{y=0}^{\infty} \left(\int_{x=0}^{\infty} g(x,y) dx \right) dy$$

although all the integrals all well defined.

(b) Set

$$h(x,y) = \frac{8xy(x^2-y^2)}{(x^2+y^2)^3}$$

for $(x,y) \neq (0,0)$, and define $h(0,0)$ however you like. Show that

$$\int_{x=0}^{\infty} \left(\int_{y=0}^{1} h(x,y) dy \right) dx \neq \int_{y=0}^{1} \left(\int_{x=0}^{\infty} h(x,y) dx \right) dy$$

although all the integrals all well defined. Hint: For $(x,y) \neq (0,0)$, we have

$$h(x,y) = \frac{\partial^2}{(\partial x)(\partial y)} \frac{x^2-y^2}{x^2+y^2}.$$

2. In thus problems, all areas and volumes are understood in the sense of high school geometry. This problem goes back to Archimedes!

Let $R > 0$. Let

$$\begin{aligned} B &= \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq R^2, \ 0 \leq z\} \\ C &= \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq R^2, \ 0 \leq z \leq R\} \\ K &= \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z^2, \ 0 \leq z \leq R\} \end{aligned}$$

(a) Draw B, C and K.

(b) Show that, for every z_0 between 0 and R, we have

$$\text{Area}(B \cap \{z = z_0\}) + \text{Area}(K \cap \{z = z_0\}) = \text{Area}(C \cap \{z = z_0\}).$$

(c) Show that $\text{Volume}(B) + \text{Volume}(K) = \text{Volume}(C)$.

3. (a) Let B be an open ball in $\mathbb{R}^d$ (say of radius r). Let $h : B \to \mathbb{R}^k$ be a C^1 map such that $(Dh)_u = 0$ at every $u \in B$. Show that h is constant. (Hint: This is meant to be easy; don't over think it.)

(b) Let $X \subset \mathbb{R}^n$ be a connected d-dimensional manifold and let U be an open set containing X. Let $h : U \to \mathbb{R}^k$ be a C^1 function such that $(Dh)_x$ restricted to $T_x X$ is 0 for all $x \in X$. Show that h restricted to X is a constant function.

4. Let V be an n-dimensional real vector space with basis $e_1, \ldots, e_n$. We define a quadratic form on V to be a function $H : V \to \mathbb{R}$ of the form

$$H\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i,j=1}^{n} Q_{ij} x_i x_j \qquad (*)$$

for some scalars Q_{ij}.

(a) Show that the definition of a quadratic form does not depend on the choice of basis e_i. In other words, show that, if $(*)$ holds and $f_1, \ldots, f_n$ is an other basis of V, then there are scalars R_{ij} such that

$$H\left(\sum_{j=1} y_j f_j\right) = \sum_{i,j=1}^{n} R_{ij} y_i y_j.$$

Let H be a quadratic form. For $\vec{u}$ and $\vec{v}$ in V, define $B(\vec{u},\vec{v}) = \frac{1}{2}\left(H(\vec{u}+\vec{v}) - H(\vec{u}) - H(\vec{v})\right)$.

(b) Show that B is a symmetric bilinear form.

(c) Show that there is a basis $v_1, \ldots, v_n$ of V such that $B(v_i, v_j) = 0$ for $i \neq j$. Show furthermore that we can choose the basis such that $B(v_i, v_i) = -1$, 0 or 1 for each i. (Hint: I know two approaches here. One uses the spectral theorem; the other mimics the Gram-Schmidt algorithm.)

(d) Let Q be a quadratic form on V. Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ be two bases of V such that $B(v_i, v_j) = B(w_i, w_j) = 0$ for $i \neq j$. Let a_+, a_0 and a_- be the number of i's for which $B(v_i, v_i)$ is > 0, $= 0$ and < 0 respectively, and let b_+, b_0 and b_- be the number of j's for which $B(w_j, w_j)$ has each sign. Show that $a_+ = b_+$, $a_0 = b_0$ and $a_- = b_-$. (Hint: If $b_- > a_-$ then $(a_+ + a_0) + b_- > n$. Use this to show there is a vector $\vec{x}$ with both $B(\vec{x},\vec{x}) \geq 0$ and $B(\vec{x},\vec{x}) < 0$.)

5. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function. For $d_1, d_2, \ldots, d_n \geq 0$, define:

$$f_{d_1 d_2 \cdots d_n} = \left(\frac{\partial}{\partial x_1}\right)^{d_1} \left(\frac{\partial}{\partial x_2}\right)^{d_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{d_n} F \Bigg|_{(x_1,\ldots,x_n)=(0,0,\ldots,0)}$$

$$P_k(x_1, \ldots, x_n) = \sum_{\substack{d_1+d_2+\cdots+d_n \leq k \\ d_1,\ldots,d_n \geq 0}} f_{d_1 d_2 \cdots d_n} \frac{x_1^{d_1}}{(d_1)!} \frac{x_2^{d_2}}{(d_2)!} \cdots \frac{x_n^{d_n}}{(d_n)!}$$

$$R_k(z) = F(z) - P_k(z).$$

(a) Show that, for any nonnegative integers $d_1, \ldots, d_n$ with $\sum d_i \leq k$, we have

$$\left(\frac{\partial}{\partial x_1}\right)^{d_1} \left(\frac{\partial}{\partial x_2}\right)^{d_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{d_n} R_k \Bigg|_{(x_1,\ldots,x_n)=(0,0,\ldots,0)} = 0.$$

(b) Show that, if $\vec{v}$ is any vector in $\mathbb{R}^n$, then

$$\frac{d^j}{(dt)^j} R_k(t\vec{v}) \Bigg|_{t=0} = 0$$

for $j \leq k$.

(c) Show that there is a constant M such that, for any vector $\vec{v} \in \mathbb{R}^n$ with $|\vec{v}| \leq 1$, we have

$$|R_k(\vec{v})| \leq M |\vec{v}|^{k+1}.$$