Problem Set 6 – due November 3

See course website for policy on collaboration.

1. Let R be a rectangle in $\mathbb{R}^n$ and let f and g be bounded functions $R \to \mathbb{R}$. Prove from the definitions:
   (a) If $f(x) \le g(x)$ for all $x \in R$, then $\underline{\int} f \le \underline{\int} g$ and $\overline{\int} f \le \overline{\int} g$.
   (b) We have
   $$\underline{\int} f + \underline{\int} g \le \underline{\int} f + g \le \overline{\int} f + g \le \overline{\int} f + \overline{\int} g.$$
   (c) If $S \supset R$ is a larger rectangle, and $f : S \to \mathbb{R}$ is a bounded function with $f(x) = 0$ for $s \in S \setminus R$, then
   $$\underline{\int}_R f = \underline{\int}_S f \text{ and } \overline{\int}_R f = \overline{\int}_S f.$$
   Whether or not you have proved them, you may assume the results of Question 1 in the rest of this (and all following) problem sets.

2. Let Q be a closed rectangle. Let R_1, R_2, $\dots$, be a sequence of open rectangles such that $Q \subseteq \bigcup R_i$. In this problem, we will show that $\mathrm{Vol}(Q) \le \sum \mathrm{Vol}(R_i)$. For any subset S of $\mathbb{R}^n$, let
   $$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}.$$
   (a) Show that there is a finite subset R_{i_1}, R_{i_2}, $\dots$, R_{i_N} of the R's with $Q \subset \bigcup_j R_{i_j}$.
   (b) Show that $\mathrm{Vol}(Q) \le \sum_{j=1}^N \mathrm{Vol}(R_{i_j})$. (Hint: Let C be a rectangle which contains all of the Q and R_{i_j}. Consider $\int_C \chi_Q$ and $\int_C \sum_{j=1}^N \chi_{i_j}$ and cite Question 1 liberally.)

3. The goal of this question is to construct a function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that
   $$\overline{\int}_{x\in[0,1]} \overline{\int}_{y\in[0,1]} f(x,y) < \overline{\int}_{(x,y)\in[0,1]\times[0,1]} f(x,y).$$
   I think part (a) of this question is the hardest; you may want to first do parts (b) and (c).
   (a) Construct a subset C of $[0,1] \times [0,1]$ such that (1) C is dense in $[0,1] \times [0,1]$ but (2) for any $x \in [0,1]$, there is at most one y such that $(x,y) \in C$.
   Let
   $$f(x,y) = \begin{cases} 1 & (x,y) \in C \\ 0 & \text{otherwise} \end{cases}.$$
   (b) Show that $\underline{\int}_{x\in[0,1]} \underline{\int}_{y\in[0,1]} f(x,y) = \overline{\int}_{x\in[0,1]} \overline{\int}_{y\in[0,1]} f(x,y) = 0$.
   (c) Show that $\overline{\int}_{(x,y)\in[0,1]\times[0,1]} f(x,y) = 1$.

   The issue pointed out in this problem is an artifact of the Riemann integral; using the Lebesgue integral, if $\int_x \int_y f(x,y)$ exists in the Lebesgue sense, then $\int_{(x,y)} f(x,y)$ exists and equals it.

4. Let A be a matrix. An *elementary row operation* is to (1) switch two rows (2) multiply a row by a nonzero scalar or (3) add a scalar multiple of one row to another. Show that, if A is an invertible matrix, then it is possible to apply elementary row operations to A to turn A into the identity. (This is a linear algebra lemma we will need next week.)

5. We introduce the following notation: Let A be an $m \times n$ matrix, let $k \le m$, n and let I be a k-element subset of $\{1, \dots, n\}$ and J a k-element subset of $\{1, 2, \dots, n\}$. Then A_{IJ} denote the matrix with rows indexed by I and columns indexed by J. Let A be a $\ell \times m$ matrix, B a $m \times n$ matrix, let $k \le \ell$, m and let L be a k-element subset of $\{1, 2, \dots, \ell\}$ and N a k-element subset of $\{1, 2, \dots, n\}$.

   Show that

$$\det(AB)_{LN} = \sum_{\substack{M \subseteq \{1,2,\dots,m\} \\ |M| = m}} \det A_{LM} B_{MN}.$$

   For clarity, we give an example:

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}\right) =$$

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \det\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \det\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} \det\begin{bmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{bmatrix} + \det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} \det\begin{bmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}.$$

6. This question introduces a Lie group we will want to consider often in the future. Set

$$I = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad K = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

   You may assume without proof the identities:

$$I^2 = J^2 = K^2 = -\mathrm{Id}_4,\ IJ = -JI = K,\ IK = -KI = -J,\ JK = -KJ = I.$$

   Let $\mathbb{H} = \mathrm{Span}_{\mathbb{R}}(\mathrm{Id}_4, I, J, K) \subset \mathrm{Mat}_{4\times 4}(\mathbb{R})$. As an abstract ring, $\mathbb{H}$ is called the **quaternions**. For a quaternion $\alpha = a\mathrm{Id}_4 + bI + cJ + dK$, we define $\overline{\alpha} = a\mathrm{Id}_4 - bI - cJ - dK$.

   (a) Check that $\overline{\alpha\beta} = \overline{\beta} \cdot \overline{\alpha}$. (The $\cdot$ on the right hand side is multiplication.)

   (b) Define $SU(2) = \{\alpha \in \mathbb{H} : \alpha\overline{\alpha} = 1\}$. Show that $SU(2)$ is a subgroup of GL_4.

   (c) Define $\mathfrak{su}(2) = \mathrm{Span}_{\mathbb{R}}(I, J, K)$. Show that $\mathfrak{su}(2)$ is the Lie algebra of SU(2).

   (d) Let $X = pI + qJ + rK \in \mathfrak{su}(2)$ and define $\theta^2 = p^2 + q^2 + r^2$. Show that

$$\exp(X) = \cos\theta \ \mathrm{Id}_4 + \tfrac{\sin\theta}{\theta} X.$$

   (e) Show that exp is injective on $\{pI + qJ + rK : p^2 + q^2 + r^2 < \pi^2\}$ and describe how exp behaves on the sphere $\{pI + qJ + rK : p^2 + q^2 + r^2 = \pi^2\}$.