Problem Set 7 – due November 10

See course website for policy on collaboration.

1. (a) Let
$$c(r,\theta,h) = (r\cos\theta, r\sin\theta, h).$$
Compute $\det Dc$ and describe when it is zero.

   (b) Let
$$s(r,\theta,\phi) = (r\cos\phi\cos\theta, r\cos\phi\sin\theta, r\sin\phi).$$
Compute $\det Ds$ and describe when it is zero.

2. In this question, we will prove that rotating coordinates in $\mathbb{R}^2$ does not change integrals. Do not simply quote the change of variables formula, as the whole point of this proof is to walk you through a special case which is missing a lot of the complications.

   Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and $R > 0$ a constant such that $f(x,y) = 0$ for $(x,y) \notin [-R,R]^2$. Let's assume that f is bounded and is continuous except on a set of measure 0, so $\int_{\mathbb{R}^2} f$ exists. (Since $f = 0$ for (x,y) large, we just have $\int_{\mathbb{R}^2} f = \int_{[-R,R]\times[-R,R]} f$.)

   (a) Let $h \in \mathbb{R}$. Show that
$$\int_{\mathbb{R}^2} f(x,y) = \int_{\mathbb{R}^2} f(x, y + hx).$$

   (b) Let a and $b > 0$. Show that
$$\int_{\mathbb{R}^2} f(ax, by) = \frac{1}{ab}\int_{\mathbb{R}^2} f(x,y).$$

   (c) Let θ be a real number. Show that the map $(x,y) \mapsto (\cos\theta x - \sin\theta y, \sin\theta x + \cos\theta y)$ can be written as a composition of maps of the forms $(x,y) \mapsto (x, y+hx)$, $(x,y) \mapsto (x+hy, y)$ and $(x,y) \mapsto (ax, by)$.

   (d) Show that
$$\int_{\mathbb{R}^2} f(x,y) = \int_{\mathbb{R}^2} f(\cos\theta x - \sin\theta y,\ \sin\theta x + \cos\theta y).$$

3. This problem provides a quick proof that $\lim_{R\to\infty} \int_{[0,R]} \frac{\sin x}{x}$ exists (where $x = 0$ may be filled in however you like.)

   (a) Integrate by parts to show, for $0 < M < N$, that
$$\left| \int_{[M,N]} \frac{\sin x}{x} \right| \leq 3/M.$$
(If you get a different constant, that's fine.)

   (b) Show that $\lim_{R\to\infty} \int_{[0,R]} \frac{\sin x}{x}$ exists.

4. In order to make proofs in this problem shorter, we will slightly broaden the definition of measure zero: We'll say that $X \subset \mathbb{R}^n$ has measure zero if, for any $\epsilon > 0$, there exist a sequence of closed bounded sets R_i such that $\mathrm{Vol}(R_i)$ exists, $X \subseteq \bigcup R_i$ and $\sum \mathrm{Vol}(R_i) < \epsilon$. (In class, we insisted that the R_i be rectangles.)

   Moreover, if a high school geometry student would know how to compute $\mathrm{Vol}(R_i)$, you may assert its value without proof.

(a) Let $f : [0,1] \to \mathbb{R}^2$ be a smooth function. Show that there is a constant $C > 0$ such that, for any $0 \leq x < y \leq 1$, the image $f([x,y])$ is contained in a ball of radius $C(y-x)$.

(b) Show that $f([0,1])$ has measure 0.

Let $g : [0,1] \times [0,1] \to \mathbb{R}^2$ be smooth. Let Z be the set of $(x,y) \in [0,1] \times [0,1]$ where $\det(Dg)_{(x,y)} = 0$. The goal of the next several parts is to show that $g(Z)$ has measure 0.

(c) Let $z \in Z$. Show that there is a nonzero vector $\vec{v} \in \mathbb{R}^2$ such that

$$\frac{\partial}{\partial x}(\vec{v} \cdot g) = \frac{\partial}{\partial y}(\vec{v} \cdot g) = 0 \text{ at } z$$

(d) Show that there is a constant $C > 0$ such that, for any $z \in Z$ and any $\epsilon \times \epsilon$ rectangle r containing z, the image $g(r)$ is contained in a shape of area $< C\epsilon^3$. Here "shape" can mean anything whose area a high school geometry student could compute.

(e) Show that $g(X)$ has measure 0.