Problem Set 8 – due November 17

See course website for policy on collaboration.

This week, we compute integrals! We also prove a useful formula for the matrix exponential.

1. In this problem, we will compute $\int_{-\infty}^{\infty} e^{-x^2} dx$.
   (a) Give exhaustions of $\mathbb{R}^2$ and $\mathbb{R}$ to show that
   $$\int_{\mathbb{R}^2} e^{-x^2-y^2} = \left(\int_{\mathbb{R}} e^{-x^2}\right)^2$$
   (b) Give another exhaustion of $\mathbb{R}^2$ to show that
   $$\int_{\mathbb{R}^2} e^{-x^2-y^2} = \lim_{R\to\infty} \int_{x^2+y^2\le R^2} e^{-x^2-y^2}.$$
   (c) Compute $\int_{x^2+y^2\le R^2} e^{-x^2-y^2}$. Hint: Try the change of variables $x = r\cos\theta$, $y = r\sin\theta$.
2. (a) Show that, for $0 < r < 1$, we have
   $$\int_{(x,y)\in[0,r]\times[0,r]} \frac{1}{1-xy} = \sum_{n=1}^{\infty} \frac{r^{2n}}{n^2}.$$
   (b) Show that
   $$\int_{(x,y)\in[0,1]\times[0,1]} \frac{1}{1-xy} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$
   (c) Make the substitution $x = u+v$, $y = u-v$ to compute that $\int_{(x,y)\in[0,1]\times[0,1]} \frac{1}{1-xy} = \frac{\pi^2}{6}$.
3. In this problem, we will compute $\lim_{M\to 0} \int_{[0,M]} \frac{\sin x}{x}$.
   (a) Show that
   $$\lim_{M\to\infty} \lim_{N\to\infty} \int_{[0,M]\times[0,N]} e^{-xy}\sin x = \lim_{M\to 0} \int_{[0,M]} \frac{\sin x}{x}.$$
   (b) Show that
   $$\lim_{N\to\infty} \lim_{M\to\infty} \int_{[0,M]\times[0,N]} e^{-xy}\sin x = \lim_{N\to\infty} \int_{[0,N]} \frac{1}{1+y^2} = \frac{\pi}{2}.$$
   We now need to justify interchanging the limits. Feel free to replace the bounds below with other bounds which do the job.
   (c) Show that
   $$\left| \int_{[0,M_1]\times[N_1,N_2]} e^{-xy}\sin x \right| \le \frac{1}{N_1}.$$
   Hint: $|\sin x| \le x$.
   (d) Show that
   $$\left| \int_{[M_1,M_2]\times[0,N_1]} e^{-xy}\sin x \right| \le \frac{3}{M_1} + \frac{1}{N_1}.$$
   Hint: I'd integrate on y first.
   (e) Show that
   $$\lim_{M\to\infty} \lim_{N\to\infty} \int_{[0,M]\times[0,N]} e^{-xy}\sin x = \lim_{N\to\infty} \lim_{M\to\infty} \int_{[0,M]\times[0,N]} e^{-xy}\sin x.$$

4. Let X and Y be $n \times n$ matrices. Recall that we write ad_X for the map $Y \mapsto [X, Y]$. The point of this problem is to prove the identity:

$$(D\exp)_X(Y)\ e^{-X} = \sum_{n=0}^{\infty} \frac{\mathrm{ad}_X^n(Y)}{(n+1)!}.$$

I have to admit I don't know a really nice proof of this one; here is an argument.

(a) Check that

$$\frac{d}{ds} e^{sX} = X e^{sX} = e^{sX} X.$$

(b) Put

$$U(s) = \frac{\partial}{\partial t} e^{s(X+tY)}\Big|_{t=0} e^{-sX}.$$

Check that

$$\frac{dU}{ds} = e^{sX} Y e^{-sX}.$$

(c) Show that

$$U(s) = \sum_{n=0}^{\infty} \frac{\mathrm{ad}_X^n(Y) s^{n+1}}{(n+1)!}$$

and deduce the identity for $(D\exp)_X(Y)$.