Problem Set 9 – due December 1

See course website for policy on collaboration.

1. (a) Write $d\left(x^2+y^2\right)$ in terms of dx and dy.
   (b) Let $g(u,v)=(e^u\cos v, e^u\sin v)$. Compute $g^*d\left(x^2+y^2\right)$ directly from your answer to the previous part.
   (c) Check that the result is equal to $d\left((e^x\cos y)^2+(e^x\sin y)^2\right)$.

2. Let γ be a circle of radius r in $\mathbb{R}^2$ centered at 0, oriented counter clockwise. Compute

$$\int_\gamma xdy - ydx.$$

3. Let V be an n-dimensional real vector space. We define a ***volume form*** on V to be a function $\alpha : \overbrace{V\times V\times\cdots\times V}^{n \text{ factors}} \to \mathbb{R}$ such that

   - α is linear as a function of each input, with the other $n-1$ inputs held fixed.
   - α is anti-symmetric in its inputs: $\alpha(\vec{v}_1,\vec{v}_2,\dots,\vec{v}_k,\vec{v}_{k+1},\dots,\vec{v}_n) = -\alpha(\vec{v}_1,\vec{v}_2,\dots,\vec{v}_{k+1},\vec{v}_k,\dots,\vec{v}_n)$.

   Define $\mathrm{Vol}(V)$ to be the vector space of volume forms. Show that $\mathrm{Vol}(V)$ is one dimensional.

4. Let V be an n-dimensional real vector space. Let $\mathrm{Vol}(V)$ be the one dimensional real vector space described in the previous problem. Let A be a bounded open set in V and let $\omega : A \to \mathrm{Vol}(V)$ be a continuous bounded function. More explicitly, we write $\omega_a(\vec{v}_1,\dots,\vec{v}_n)$ for $a\in A$ and $\vec{v}_1, \dots, \vec{v}_n$ in V. The goal of this problem is to indicate how to define $\int_A \omega$, up to sign.

   Let $e_1, e_2, \dots, e_n$ be a basis for V and define $\epsilon : \mathbb{R}^n \to V$ by $\epsilon(x_1,\dots,x_n)=\sum x_i e_i$. Let $f_1, f_2, \dots, f_n$ be another basis for V and define $\phi:\mathbb{R}^n\to V$ by $\phi(y_1,\dots,y_n)=\sum y_i f_i$.

   Show that
$$\left|\int_{x\in\epsilon^{-1}(A)} \omega_{\epsilon(x)}(e_1,\dots,e_n)\right| = \left|\int_{y\in\phi^{-1}(A)} \omega_{\phi(y)}(f_1,\dots,f_n)\right|.$$

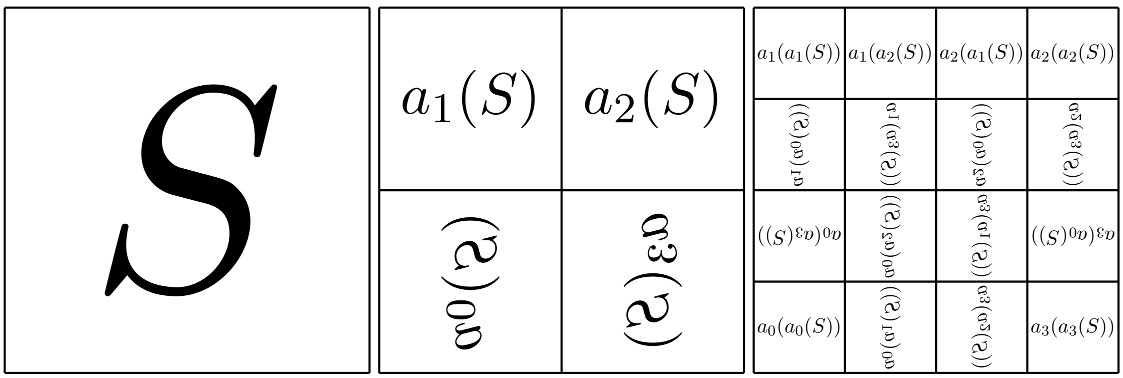
5. In this question we will finally get around to something we should have proved a while ago: If Y is a matrix with $|Y|<1$ then $\log(1+Y)=\sum_{n=1}^\infty \frac{(-1)^{n-1}Y^n}{n}$. Here $|Y|=\sqrt{\sum_{i,j=1}^n Y_{ij}^2}$. Recall that $|YZ|\le|Y|\cdot|Z|$ and $|Y+Z|\le|Y|+|Z|$.

   For an $n\times n$ matrix Y with $|Y|<1$, define $L(\mathrm{Id}_n+Y)=\sum_{n=1}^\infty \frac{(-1)^{n-1}Y^n}{n}$.
   (a) Check that the sum $\sum_{n=1}^\infty \frac{(-1)^{n-1}Y^n}{n}$ converges for $|Y|<1$.
   (b) Let Y and Z commute. Show that
$$(DL)_{\mathrm{Id}_n+Y}(Z) = (\mathrm{Id}_n+Y)^{-1}Z.$$
   (c) Show that, for t close enough to zero for $L(e^{tX})$ to be defined, we have $\frac{d}{dt}L(e^{tX})=X$. Deduce that, for X in an open ball around 0, we have $L(e^X)=X$.
   (d) Show that, for Y in an open ball around 0, we have $L(\mathrm{Id}_n+Y)=\log(\mathrm{Id}_n+Y)$.

6. This question constructs a continuous surjection $h : [0,1] \to [0,1]^2$. This is a construction of Hilbert. We'll abbreviate $[0,1]^2$ to S. Define maps a_0, a_1, a_2, $a_3 : S \to S$ as follows:

$$\begin{array}{rcll} a_0(x,y) & = & (y/2, & x/2) \\ a_1(x,y) & = & (x/2, & y/2+1/2) \\ a_2(x,y) & = & (x/2+1/2, & y/2+1/2) \\ a_3(x,y) & = & (-y/2+1, & -x/2+1/2) \end{array}$$

As an aid to visualization, the diagrams below show the images of all maps a_i and $a_i \circ a_j$, with the orientation of the text indicating the geometric transformation. I encourage you to print out several copies and write on them to help understand them.



(a) Let i_1, i_2, i_3, $\ldots$ be any sequence of elements of $\{0,1,2,3\}$. Show that

$$\lim_{N\to\infty} a_{i_1}(a_{i_2}(a_{i_3}(\cdots(a_{i_N}(0,0))\cdots)))$$

exists. Define this limit to be $f(i_1, i_2, i_3, \ldots)$.

(b) Let $x \in [0,1]$ and write x in base 4 as $\sum_{a=1}^{\infty} i_a 4^{-a}$. Define $h(x) = f(i_1, i_2, i_3, \ldots)$. Show that, if x can be written in base 4 in two ways (such as $0.03333\cdots = 0.10000\cdots$), then the value of h doesn't depend on which expression you use. Show that $h : [0,1] \to S$ is continuous.

(c) Show that $h(0) = (0,0)$, $h(1/3) = (0,1)$, $h(2/3) = (1,1)$ and $h(1) = (1,0)$. Compute $h(k/12)$ for $0 \le k \le 12$.

(d) Show that, for $0 \le p \le 3 \cdot 4^k$, the values of $h(p/(3 \cdot 4^k))$ cover all the points of the form $(q/2^k, r/2^k)$ for $0 \le q, r \le 2^k$.

(e) Show that the image of h is closed, and hence is all of S. Hint: $[0,1]$ is compact.