Finishing the proof of the inverse function

The goal of this worksheet is to prove the inverse function theorem:

Theorem. *Let V and W be finite dimensional vector spaces. Let A be an open subset of V and let $f : A \to W$ be a C^1 function. Let $a \in A$, set $f(a) = b$, and suppose the linear map $(Df)(a)$ is bijective. Then there are open sets $A' \ni a$ in A and $B' \ni b$ in W such that f is a bijection A' to B' and f^{-1} is differentiable on B'.*

By our previous work, we may (and do) shrink A such that there is a constant $c > 0$ such that

$$|f(x_1) - f(x_2)| \geq c|x_1 - x_2|$$

for x_1, $x_2 \in A$. Also, by our previous work, we may (and do) shrink A such that, for any open $U \subseteq A$, the set $f(U)$ is open.

Problem 1. Show that f is injective on A.

Problem 2. Show that there is a neighborhood $B' \ni b$ such that $B' \subseteq f(A)$.

Problem 3. Show that there is neighborhood $A' \ni a$ in A such that f is a bijection $A' \to B'$.

So it now makes sense to talk about $f^{-1}(y)$ for $y \in B'$. We define $g : B' \to A'$ to be the inverse of f.

Problem 4. Let y_1 and $y_2 \in B'$ and let $x_i = g(y_i)$. Show that there is a constant c_2 such that

$$|x_1 - x_2| \leq c_2|y_1 - y_2|.$$

Deduce as a corollary that g is continuous.

For brevity, set $E = D(f)(a)$ and recall that E is assumed invertible.

Problem 5. For h small enough that $b + h \in B'$, show that

$$\frac{|g(b+h) - g(b)|}{|h|} \leq c_2.$$

For h as above, put $k(h) = g(b + h) - g(b)$.

Problem 6. Show that, as $h \to 0$, the function $k(h)$ goes to 0 as well.

Problem 7. Show that

$$\begin{aligned} \frac{|g(b+h)-g(b)-E^{-1}(h)|}{|g(b+h)-g(b)|} &= \frac{|k(h)-E^{-1}(f(a+k(h))-f(a))|}{|k(h)|} \\ &= E^{-1}\frac{|Ek(h)-f(a+k(h))-f(a))|}{|k(h)|} \end{aligned}.$$

Problem 8. Show that, as $h \to 0$, we have

$$\frac{|g(b+h) - g(b) - E^{-1}(h)|}{|g(b+h) - g(b)|} \to 0.$$

Problem 9. Prove the theorem by proving g is differentiable at b with derivative E^{-1}. In other words, as $h \to 0$, we have

$$\frac{|g(b+h) - g(b) - E^{-1}(h)|}{|h|} \to 0.$$