INTEGRATION OF VECTOR VALUED FUNCTIONS

Let V be a finite dimensional vector space and $g : \mathbb{R} \to V$ a function. The point of this note is to discuss the meaning of the integral $\int_a^b g(t)dt$.

We have already seen that $g'(t)$ can be defined as $\lim_{h\to 0} \frac{g(t+h)-g(t)}{h}$ or can be defined coordinate by coordinate: If $g(t) = (g_1(t),\ g_2(t),\ \cdots,\ g_n(t))$ then $g'(t) = (g_1'(t),\ g_2'(t),\ \ldots,\ g_n'(t))$. See Problem 5 on Problem Set 0. Moreover, this does not depend on choice of basis or on the choice of norm on V (since they are all equivalent). In terms of our sophisticated notion of Dg as a linear map from $\mathbb{R}$ to V, we have

$$(Dg)_t(1) = g'(t).$$

This paragraph is meant to look very close to what you are used to from earlier classes: We define a ***partition of*** $[a, b]$ to be a sequence $a = x_0 < x_1 < \cdots < x_N = b$. We define the ***mesh*** of a partition to be $\max_{1\le i\le N} x_i - x_{i-1}$. We define a ***tagging*** of a partition $(x_0, x_1, \ldots, x_n)$ to be a choice t_i in each interval $[x_{i-1}, x_i]$. Given a tagged partition, the Riemann sum is defined to be

$$\sum_{i=1}^{N} (x_i - x_{i-1})g(t_i).$$

We say that

$$\int_a^b g(t)dt = \vec{w}$$

if, for any $\epsilon > 0$ there exists a $\delta > 0$ such that, if $(x_0, x_1, \ldots, x_n)$ is any partition with mesh $< \delta$ and $(t_1, \ldots, t_N)$ is any tagging of that partition, then

$$\left| \sum_{i=1}^{N} (x_i - x_{i-1})g(t_i) - \vec{w} \right| < \epsilon.$$

Clearly, a function has at most one integral and (because all norms on V are equivalent) does not depend on what norm is used. Similarly to Problem 5 on Problem Set 0, if one chooses a basis for V, then one can prove that a function is integrable if and only if each component is integrable, in which case the integral does not depend on the choice of basis.

If $f : [a, b] \to V$ is differentiable, we have

$$\int_a^b f'(t)dt = f(b) - f(a).$$

One can either adapt standard proofs of the fundamental theorem of calculus or just work coordinate by coordinate.