IBL: Maximal Tori

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Notation and Goals

Let $T \subset \mathrm{GL}_n(\mathbb{C})$ be a compact abelian Lie Group and let $\mathfrak{t} \subset \mathrm{Mat}_{n\times n}(\mathbb{C})$ be its Lie algebra. Define D to be the abelian Lie Group

$$D := \left\{ \begin{bmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{bmatrix} : \theta_1, \dots, \theta_n \in \mathbb{R} \right\}$$

which has Lie algebra

$$\mathfrak{d} := \left\{ \begin{bmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{bmatrix} : \theta_1, \dots, \theta_n \in \mathbb{R} \right\}$$

Last week we showed that $\exists S \in \mathrm{GL}_n(\mathbb{C})$ such that

$$S\mathfrak{t}S^{-1} \subset \mathfrak{d} \quad \text{and} \quad STS^{-1} \subset D$$

That is, that we can "diagonalize" any compact abelian Lie group T.

Now, let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a **compact** Lie group. We wish to find $S \in \mathrm{GL}_n(\mathbb{C})$ such that $SGS^{-1} \cap D$ is "as large as possible". First, let us ensure that this construct is even a Lie subgroup of G. We do this with the following Lemma:

Lemma 1 (Intersection of Lie Subgroups)**.** *Let G and H be Lie subgroups of GL_n (the proof is equivalent whether we are over $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$) with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Then $G \cap H$ is a Lie group with Lie algebra $\mathfrak{g} \cap \mathfrak{h}$.*

Proof. First, we note that the intersection of two subgroups of a group is itself a subgroup, so $G \cap H$ is a subgroup of GL_n. It remains to show that $G \cap H$ is a manifold and $\mathfrak{g} \cap \mathfrak{h}$ is its tangent space at $\mathrm{Id}_n \in \mathrm{GL}_n$. Therefore, fix $z \in G \cap H$. We split this proof into two parts:

- Let $z = \mathrm{Id}$.
  Note by previous IBL that exp is locally invertible near Id between any Lie group and its Lie algebra. Therefore, $\exists U_g, V_g, U_h, V_h$ open such that

$$0 \in U_g \subset \mathfrak{g}, \quad \mathrm{Id} \in V_g \subset G, \quad \exp : U_g \to V_g \text{ is a homeomorphism.}$$

$$0 \in U_h \subset \mathfrak{h}, \quad \mathrm{Id} \in V_h \subset H, \quad \exp : U_h \to V_h \text{ is a homeomorphism.}$$

Note that by subspace topologies, $U_g \cap U_h$ is open in $\mathfrak{g} \cap \mathfrak{h}$ and contains 0, $V_g \cap V_h$ is open in $G \cap H$ and contains Id, and since exp is continuously invertible on all of U_g and U_h, its restriction to $U_g \cap U_h$ is also a homeomorphism. Finally,

$$x \in U_g \cap U_h \implies \exp(x) \in V_g \text{ and } \exp(x) \in V_h \implies \exp(x) \in V_g \cap V_h$$

so

$$\exp : U_g \cap U_h \to V_g \cap V_h \text{ is a homeomorphism.}$$

Finally, since $\mathfrak{g}$ and $\mathfrak{h}$ are linear subspaces, their intersection is a linear subspace, so exp is a homeomorphic patch onto an open subset of $G \cap H$ containing Id.

- Now, take $z \in G \cap H$.
  Note that as a function, multiplication by z is a homeomorphism on $G \cap H$ so it takes open sets to open sets. Therefore, using the previous result, $\exists U_0 \subset \mathfrak{g} \cap \mathfrak{h}$ open containing 0, $V_{\mathrm{Id}} \subset G \cap H$ open containing Id, such that exp is a homeomorphism from U_0 to V_{Id}. Therefore,

  $$U_0 \xrightarrow{\exp} V_{\mathrm{Id}} \xrightarrow{\cdot z} zV_{\mathrm{Id}}$$

  is a composition of homeomorphisms, and thus is a homeomorphism from $U_0 \in \mathfrak{g} \cap \mathfrak{h}$ to zV_{Id} which contains $z = z \cdot \mathrm{Id}$.

Thus $G \cap H$ is a manifold with tangent space at Id given by $\mathfrak{g} \cap \mathfrak{h}$ (recall that $[D \exp]_0 = \mathrm{Id}_{n^2}$). Thus $G \cap H$ is a Lie subgroup with Lie algebra $\mathfrak{g} \cap \mathfrak{h}$. □

Remark 1. *Note that if G and H are compact, then $G \cap H$ is compact because the intersection of compact sets is compact. Similarly, if either G or H is abelian, then $G \cap H$ is abelian because it is a subgroup of both G and H.*

Thus we know that $T = SGS^{-1} \cap D$ is a compact, abelian Lie subgroup of G. We wish to find a maximal choice for T.

Problems

As above, let G be a **compact** Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ and let $\mathfrak{g}$ be its Lie algebra. We define a ***torus*** T of G to be a compact connected abelian Lie subgroup of G. We say T is a ***maximal torus*** of G if it is not contained in any torus of G of higher dimension.
Let T be a torus of G and let $\mathfrak{t}$ be its Lie algebra. Define $V_{\mathfrak{t}} \subset \mathfrak{g}$ by

$$V_{\mathfrak{t}} := \{v \in \mathfrak{g} : [\theta, v] = 0 \,\forall\, \theta \in \mathfrak{t}\}$$

Problem 5. *Show that $V \supset \mathfrak{t}$.*

Proof. By definition, T is abelian. So, from IBL on 2/2/19, all elements of $\mathfrak{t}$ commute. Thus

$$[\theta, v] = \theta v - v\theta = 0$$

for all $\theta, v \in \mathfrak{t}$, so $\mathfrak{t} \subset V$. □

Problem 6. *Suppose that $V = \mathfrak{t}$. Show that T is maximal.*

Proof. Suppose T is not maximal; so there exists torus T' such that $T \subset T'$. We then have $\mathfrak{t} \subset \mathfrak{t}'$ with $\dim \mathfrak{t} < \dim \mathfrak{t}'$. So, there exists $u \in \mathfrak{t}' \setminus \mathfrak{t}$. Now, for all $\theta \in \mathfrak{t}$, we have $\theta \in \mathfrak{t}'$; then since T' is abelian, we conclude that $[u, \theta] = 0$. Thus $u \in V$. But $V = \mathfrak{t}$ and $u \notin \mathfrak{t}$, so we have a contradiction. □

Our goal now is to prove the converse of the previous problem: If $V \supsetneq \mathfrak{t}$, then T is not maximal. Let $v \in V$ with $v \notin \mathfrak{t}$. Let U be the vector space spanned by $\mathfrak{t}$ and v.

Problem 7. *Show that there is some $s \in \mathrm{GL}_n\mathbb{C}$ such that $U \subseteq s\mathfrak{d}s^{-1}$. (Recall that G is compact.)*

Proof. First, note that $v \in \mathfrak{g}$ and $\mathfrak{t} \subset \mathfrak{g}$, so since $\mathfrak{g}$ is a vector space, we have $U \subset \mathfrak{g}$. So, $\exp(U) \subset G$. Now, we recall that G is compact; thus G is bounded, so $\exp(U)$ is bounded. So, by IBL 2/2/18 Problem 1, every element of U is diagonalizable with imaginary eigenvalues. Additionally, $[u_1, u_2] = 0$ for all $u_1, u_2 \in U$, by the bilinearity of bracket and the definition of V. Let $u_1, \ldots, u_n$ be a basis for U. Then all the u_i commute and are diagonalizable; so by IBL 2/2/18 Problem 3, there exists change of basis that diagonalizes all u_i at once. Since the u_i form a basis, this diagonalizes all of U. That is, there exists $s \in \mathrm{GL}_n\mathbb{C}$ such that $U \subseteq s\mathfrak{d}s^{-1}$. □