396 IBL 1

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Jordan Normal Forms

1. Fix $\lambda \in V$. Suppose that $\lambda \in I_n$, meaning that there is a $v \in V$ such that $A^n(v) = \lambda$. This means that $\lambda = A^{n-1}(A(v))$, so $\lambda \in I_{n-1}$. So, $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$. Now, fix $\gamma \in V$, and suppose that $\gamma \in N_m$, meaning that $A^m(\gamma) = 0$. Well, this means that $A(A^m(\gamma)) = A^{m+1}(\gamma) = 0$, so $\gamma \in N_{m+1}$. So, $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots$.

2. To show I and N are subspaces of V, we must show that they are subgroups with respect to vector addition, and that they play nicely with scalar multiplication. This may be reduced to showing that each is nonempty and if α, β are in our set, and $a \in F$, that $a\alpha + \beta$ is in our set. Note that $0 \in I \cap N$, so both are nonempty. Now fix α, $\beta \in I$. Then $\alpha = A^j v$ and $\beta = A^j w$ $\forall j \in \mathbb{N}$ for some $v, w \in V$. Observe: $a\alpha + \beta = aA^j v + A^j w = A^j(av) + A^j w = A^j(av + w) \in I$. Now fix α, $\beta \in N$. Then there are natural numbers p, q such that $A^p\alpha = 0$ and $A^q\beta = 0$. Without loss of generality, suppose $p < q$. Then $A^q\alpha = 0$ as well. Let's show $a\alpha + \beta \in N$. Well, $A^q(a\alpha + \beta) = aA^q(\alpha) + A^q(\beta) = a0 + 0 = 0$, so $a\alpha + \beta \in N_q \subset N$.

3. Fix $x \in I$. Hence, $x \in I_j \, \forall j \in \mathbb{N}$. Then $Ax \in I_{j+1} \, \forall j$. So $AI \subset I$. Now fix $y \in N$; by our definition of N, there exists j such that $y \in N_j$. So $A^j y = 0 \Rightarrow 0 = A(A^j y) = A^{j+1} y = A^j(A(y)) \Rightarrow A(y) \in N_j \subset N$.

4. Look at this:
$$I = \cap_{j=1}^{\infty} A(I_j) = \cap_{j=1}^{\infty} I_{j+1} \subset A(\cap_{j=1}^{\infty} I_j) = A(I) \subset I.$$
Hence, $A : I \to I$ is surjective and thus an isomorphism. Now, fix some $v \in N$. So, there is a $j \in \mathbb{N}$ such that $v \in N_j$. This means that $A^j(v) = 0$, so $A : N \to N$ is nilpotent.

   To conclude, let's show that $V = I \oplus N$. First, I'd like to justify that $I = I_j$ for some j. Note that the sequence $(\dim(I_n))$ is a non-increasing sequence of non-negative integers. Thus, it must converge to some non-negative integer. So, we know that $\dim(I) = \dim(I_j)$ for some j, and because $I \subseteq I_j$, this is enough to conclude that $I = I_j$. Because $A : I \to I$ is invertible, $A : I_j \to I_j$ is invertible. Let the inverse of A restricted to I_j be known as B. Now, fix some $v \in V$. The result will be shown if I can produce a unique element in $I \oplus N$ which is equivalent to v. Well, $A^j(v) \in I_j$, so $B(A^j(v)) \in I_j$. Let $B(A^j(v)) = w$. Now, let's look at $A^j(v - w)$.
$$A^j(v - w) = A^j(v) - A^j(w) = A^j(v) - A^j(B(A^j(v))) = A^j(v) - A^j(v) = 0$$
So we see that $v - w \in N_j \subseteq N$. Finally, observe that $v = w + (v - w) \in I \oplus N$.