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Let V be a finite dimensional vector space over a field k and $A : V \to V$ a linear map. Suppose that there are $\lambda_1, \lambda_2, \ldots, \lambda_r$ in k and positive integers $a_1, a_2, \ldots, a_r$ such that $(\prod_{i=1}^{r}(A - \lambda_i \mathrm{Id})^{a_i}) = 0$. Our goal for the next several problems is to prove that V decomposes as $V_1 \oplus V_2 \oplus \ldots V_r$, where A maps each V_i to itself and $A - \lambda_i Id$ is nilpotent on V_i. Our proof is by induction on r.

5. Explain why the base case, $r = 1$, is trivial.
   *Proof:*In the case $r = 1$, $(A - \lambda_1 \mathrm{Id})^{a_1} = 0$. Set $V_1 = V$. Clearly A maps V_1 into itself and $A - \lambda_1 \mathrm{Id}$ is nilpotent because $(A - \lambda_1)^{a_1} = 0$. □

6. Now suppose that $r \geq 2$. From our previous results, we know that we can write $V = I \oplus N$, where $A - \lambda_r Id : I \to I$ is invertible and $A - \lambda_r Id : N \to N$ is nilpotent. Show that $(\prod_{i=1}^{r-1}(A - \lambda_i \mathrm{Id})^{a_i}) = 0$ on the subspace I. (Notice that the top index is $r - 1$).
   Proof: We know that $\prod_{i=1}^{r-1}(A - \lambda_i \mathrm{Id})^{a_i}(A - \lambda_r)^{a_r} = \prod_{i=1}^{r}(A - \lambda_i \mathrm{Id})^{a_i} = 0$ but $A - \lambda_r \mathrm{Id} : I \to I$ is invertible. In particular, $(A - \lambda_r \mathrm{Id})^{a_r} : I \to I$ is invertible so that $(A - \lambda_r \mathrm{Id})^{a_r}(I) = I$. Therefore,

$$0 = \left(\prod_{i=1}^{r}(A - \lambda_i \mathrm{Id})^{a_i}\right)(I) = \left(\prod_{i=1}^{r-1}(A - \lambda_i \mathrm{Id})^{a_i}\right)((A - \lambda_r \mathrm{Id})^{a_r}(I)) = \left(\prod_{i=1}^{r-1}(A - \lambda_i \mathrm{Id})^{a_i}\right)(I)$$

□

7. Finish the proof: show that V decomposes as $V_1 \oplus V_2 \oplus ... \oplus V_r$ where A maps each V_i to itself and $A - \lambda_i Id$ is nilpotent on V_i.
   Proof: We can decompose V into $V = I_{A-\lambda_r} \oplus N_{A-\lambda_r}$, where $A - \lambda_r$ nilpotent on $N_{A-\lambda_r}$. Note that this means that $N_{A-\lambda_r} = V_{\lambda_r}$, the λ_r-eigenspace of V with respect to A. So $V = I_{A-\lambda_r} \oplus V_{\lambda_r}$. By number 6, $\left(\prod_{i=1}^{r-1}(A - \lambda_i \mathrm{Id})^{a_i}\right)(I)$ is identically 0 on $I_{A-\lambda_r}$, which means that by the inductive hypothesis, we can decompose $I_{A-\lambda_r} = I_{A-\lambda_{r-1}} + N_{A-\lambda_{r-1}}$. To conclude: repeating this process recursively from λ_{r-2} to λ_1 justifies the "$\ldots$":

$$\begin{aligned} V &= I_{A-\lambda_r} \oplus N_{A-\lambda_r} = I_{A-\lambda_r} \oplus V_{\lambda_r} \\ &= I_{A-\lambda_{r-1}} \oplus N_{A-\lambda_{r-1}} \oplus V_{\lambda}r = I_{A-\lambda_{r-1}} \oplus V_{\lambda_{r-1}} \oplus V_{\lambda}r = \ldots \\ &= V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r} \end{aligned}$$

□