Jordan Normal Form Day 3

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Let W be a finite dimensional vector space over a field k and let $B : W \to W$ be nilpotent (meaning $B^m = 0$ for some m). Let $n = \dim W$. It this problem, we will show that we can find $j_1, j_2, \ldots, j_s$ with $j_1 + j_2 + \cdots + j_s = n$ and a basis e^p_q, with $1 \leq p \leq s$ and $1 \leq q \leq j_p$ such that B acts on this basis by

$$\begin{aligned}
&0 \leftarrow\!\shortmid e^1_1 \leftarrow\!\shortmid e^1_2 \leftarrow\!\shortmid e^1_3 \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid e^1_{j_1} \\
&0 \leftarrow\!\shortmid e^2_1 \leftarrow\!\shortmid e^2_2 \leftarrow\!\shortmid e^2_3 \leftarrow\!\shortmid e^2_4 \leftarrow\!\shortmid e^2_5 \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid e^2_{j_2} \\
&\vdots \\
&0 \leftarrow\!\shortmid e^s_1 \leftarrow\!\shortmid e^s_2 \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid e^s_{j_s}
\end{aligned} \qquad (*)$$

Our proof is by induction on n. The base case $n = 0$ is trivial, so we assume $n > 0$. Let $\overline{W}$ be the image of W.

Problem 8: Show that $\dim \overline{W} < \dim W$.

Proof.

Since B is nilpotent, B is not invertible.Then $\dim \overline{W} < \dim W$.

□

Problem 9: Show that B maps $\overline{W}$ to itself.

Proof.

$\overline{W} = B(W)$ is the subspace of W, i.e. $\overline{W} \subset W$.

Then $B(\overline{W}) \subset B(W) = \overline{W}$. So B maps $\overline{W}$ to itself.

□

By induction, we can find $\bar{j}_1, \bar{j}_2, \ldots, \bar{j}_{\bar{s}}$ and a basis e_q^p for $\overline{W}$ as above.

Problem 10: Show that, for each p, you can find a vector $e^p_{\bar{j}_p+1}$ in W such that $Be^p_{\bar{j}_p+1} = e^p_{\bar{j}_p}$.

Proof.

For each p, since $e^p_{\bar{j}_p} \in \overline{W}$ and B maps W to $\overline{W}$, there exists $e^p_{\bar{j}_p+1} \in W$ such that $Be^p_{\bar{j}_p+1} = e^p_{\bar{j}_p}$.

□

So we now have vectors obeying $(*)$, but they aren't a basis yet.

Problem 11: Show that the vectors e_q^p which you have constructed so far are linearly independent.

Proof. Suppose

$$\sum_p \sum_{q=1}^{\bar{j}_p+1} c_q^p e_q^p = 0$$

for some scalars c_q^p, then

$$B(\sum_p \sum_{q=1}^{\bar{j}_p+1} c_q^p e_q^p) = \sum_p \sum_{q=1}^{\bar{j}_p} c_{q+1}^p e_q^p = 0.$$

Since these e_q^p form a basis of $\overline{W}$ by construction, $c_q^p = 0$ for $q > 1$. Therefore, back to the previous equation, we get

$$\sum_p c_1^p e_1^p = 0.$$

Again, since e_1^p are linearly independent, $c_1^p = 0$. Thus all the coefficients are 0, so e_q^p are linearly independent. □

Choose some additional vectors $f_1, f_2, \ldots, f_t$ such that the e_q^p you have already constructed, together with $f_1, f_2, \ldots, f_t$ form a basis for W.

Problem 12: Explain why there are constants c_q^p (dependent on r) such that

$$Bf_r = \sum_{p=1}^{s} \sum_{q=1}^{\bar{j}_p} c_q^p e_q^p.$$

Proof.

From **Problem 11**, we know e_q^p is a basis for $\overline{W}$.

Since B maps W to $\overline{W}$, $Bf_r \in \overline{W}$. Then Bf_r can be written as the linear combination of the basis e_q^p, i.e.

$$Bf_r = \sum_{p=1}^{s} \sum_{q=1}^{\bar{j}_p} c_q^p e_q^p.$$

□

Put

$$g_r = f_r - \sum_{p=1}^{s} \sum_{q=1}^{\bar{j}_p} c_q^p e_{q+1}^p.$$

Problem 13: Show that the e_q^p, together with g_r, is the desired basis.

Proof.

We already know e_q^p obeying $(*)$, then we just need to show $Bg_r = 0$.

$$Bg_r = Bf_r - \sum_{p=1}^{s} \sum_{q=1}^{\bar{j}_p} c_q^p B e_{q+1}^p$$

We also have $Be_{\bar{j}_p+1}^p = e_{\bar{j}_p}^p$

$$\sum_{p=1}^{s} \sum_{q=1}^{\bar{j}_p} c_q^p B e_{q+1}^p = \sum_{p=1}^{s} \sum_{q=1}^{\bar{j}_p} c_q^p e_q^p$$

Then $Bg_r = Bf_r - Bf_r = 0$.

Besides, $f_1, ..f_r$, with e_q^p, is a basis for W. Linear combination of basis is still a basis. So e_q^p together with g_r is the desired basis.

□