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1 THE UNITARY GROUP

Definition. *Let V be a complex vector space. A* ***sesquilinear form*** *is a map $*: V \times V \to \mathbb{C}$ obeying:*

$$\vec{u} * (\vec{v} + \vec{w}) = \vec{u} * \vec{v} + \vec{u} * \vec{w}$$

$$(\vec{u} + \vec{v}) * \vec{w} = \vec{u} * \vec{w} + \vec{v} * \vec{w}$$

$$\lambda(\vec{u} * \vec{v}) = (\bar{\lambda}\vec{u}) * \vec{v} = \vec{u} * (\lambda\vec{v})$$

Definition. *A sesquilinear form is called* ***Hermitian*** *if it obeys the following:*

$$\vec{u} * \vec{v} = \overline{\vec{v} * \vec{u}}$$

Note that this implies that $\vec{v} * \vec{v} \in \mathbb{R}$.

Definition. *A Hermitian form is called* ***positive definite*** *if, for all nonzero $\vec{v}$, we have $\vec{v} * \vec{v} \neq 0$*

If we identify V with $\mathbb{C}^n$, then sequilinear forms are of the form $\vec{v} * \vec{w} = \vec{v}^\dagger Q \vec{W}$ for an arbituary $Q \in Mat_{n \times n}(\mathbb{C})$, the Hermitian condition says that $Q = Q^\dagger$, and the positive definite condition says that Q is positive definite Hermitian. The standard Hermitian form on $\mathbb{C}^n$ is $\vec{v}^\dagger \vec{w}$.

Let V be a finite complex vector space equipped with a posiive definite Hermitian form $*$. We define $A : V \to V$ to be unitary if $(A\vec{v}) * (A\vec{w}) = \vec{v} * \vec{w}$, $\forall \vec{v}, \vec{w} \in V$. We will show taht V has a $*$-orthonormal basis of eigenvectors, whose eigenvalues are all of the form $e^{i\theta}$. Our proof is by induction on n.

Let λ be a complex eigenvalue of A, with eigenvector $\vec{v}$.

Problem 1.1. *Show that $|\lambda| = 1$*

Proof.

$$\vec{v} * \vec{v} = A\vec{v} * A\vec{v} = (\lambda\vec{v}) * (\lambda\vec{v}) = \lambda\bar{\lambda}(\vec{v} * \vec{v})$$

Well, since $\vec{v}$ is an eigenvector, it is not zero. Since * is positive definite, we know that $\vec{v} * \vec{v} \neq 0$ Well this implies that $1 = \lambda\bar{\lambda}$ So we have $||\lambda|| = \sqrt{\lambda\bar{\lambda}} = 1$

Remark: This shows that $\lambda = e^i\theta$ for some $\theta \in \mathbb{R}$ □

Problem 1.2. *Show that A takes $\vec{v}^{\perp} := \{\vec{w} : \vec{v} * \vec{w} = 0\}$ to itself.*

Proof. We want to show that $\forall \vec{w} \in \vec{v}^{\perp}$, $(A\vec{w}) * \vec{v} = 0$. Fix $\vec{w} \in \vec{v}^{\perp}$. Note that from the previous problem, we have that $\lambda \neq 0$. Thus $(A\vec{w}) * \vec{v} = 0 \iff \lambda(A\vec{w}) * \vec{v}$. Well $\lambda(A\vec{w}) * \vec{v} = (A\vec{w}) * (\lambda\vec{v}) = (A\vec{w}) * (A\vec{v})$. Using the fact that A is unitary, we have that $\lambda(A\vec{w}) * \vec{v} = \vec{w} * \vec{v} = 0$ by definition. Thus $(A\vec{w}) * \vec{v} \in \vec{v}^{\perp}$ □

Problem 1.3. *Explain why we are done*

Proof.
Claim: $V = \mathbb{C}\vec{v} \oplus \vec{v}^{\perp}$

Its enough to show that $\mathbb{C}\vec{v} \cap \vec{v}^{\perp}$ is trivial and that $\mathbb{C}\vec{v} + \vec{v}^{\perp}$ spans V.

Well, $\mathbb{C}\vec{v} \cap \vec{v}^{\perp} = \{0\}$ since $\vec{v} * \vec{v} \neq 0$ if $\vec{v} \neq 0$ by positive definiteness.

Fix $\vec{x} \in V$. Let $\vec{p} = (\vec{v} * \vec{x})(\vec{v} * \vec{v})^{-1}\vec{v} \in \mathbb{C}\vec{v}$. Let $\vec{q} = \vec{x} - \vec{p}$. Note that $\vec{v} * \vec{q} = \vec{v} * (\vec{x} - \vec{p}) = \vec{v} * \vec{x} - \vec{v} * (\vec{v} * \vec{x})(\vec{v} * \vec{v})^{-1}\vec{v} = \vec{v} * \vec{x} - (\vec{v} * \vec{x})(\vec{v} * \vec{v})^{-1}(\vec{v} * \vec{v}) = \vec{v} * \vec{x} - \vec{v} * \vec{x} = 0$. So $q \in \vec{v}^{\perp}$. Well $\vec{x} = \vec{p} + \vec{q} \in \mathbb{C}\vec{v} + \vec{v}^{\perp}$

This concludes the proof of the claim. We will use induction to prove that we have *-orthonormal basis of eigen vectors.

Base case) $dimV = 1$. Well, then our matrix A is 1 by 1. This is already diagonalized.

Inductive step) Suppose that for all unitary maps between n dimensional $\mathbb{C}$ vector spaces, we can find an orthonormal eigen basis.

Now suppose that A is an unitary map from V to V where V is a $n+1$ dimensional $\mathbb{C}$ vector space. Consider the characteristic polynomial of A. We know that it has degree of $n+1$. Since $\mathbb{C}$ is algebraically closed, we can find $\lambda \in \mathbb{C}$, which is a zero the characteristic polynomial. This means $det(A - \lambda) = 0$. So $ker(A - \lambda)$ is not trivial. Fix $\vec{v} \in ker(A - \lambda) - \{0\}$. WOLOG scale $\vec{v}$ such that it is an unit vector. Note that $\vec{v}$ and λ are the corresponding eigen value and vector. From our claim, we know that we can write $V = \mathbb{C}\vec{v} \oplus \vec{v}^{\perp}$. From previous problem, we know that A takes $\vec{v}^{\perp}$ to $\vec{v}^{\perp}$. Well, since that A is unitary, its restriction to $\vec{v}^{\perp}$ is also unitary. So A is an unitary map from $\vec{v}^{\perp}$ to $\vec{v}^{\perp}$ and $dim(\vec{v}^{\perp}) = n + 1 - \dim(\mathbb{C}\vec{v}) = n$. By inductive hypothesis, we can find an orthonormal eigen basis β of A in $\vec{v}^{\perp}$. Well since $\vec{v}$ is perpendicular to every element in $\beta \subset \vec{v}^{\perp}$, $\alpha = \beta \cup \{\vec{v}\}$ is a basis. Furthermore, they are all orthogonal and are an eigen vector. □

2 THE ORTHOGONAL GROUP

On this page we make sure to get a clear record of a result which was done confusingly in homework: A real orthogonal matrix can be put into block diagonal form where the blocks are $\begin{bmatrix} \pm 1 \end{bmatrix}$ and $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Let V be a finite dimensional real vector space equipped with a positive definite symmetric bilinear form $\cdot$: $V \times V \to \mathbb{R}$. We can then extend $\cdot$ to a symmetric bilinear form $\cdot$: $(V \otimes \mathbb{C}) \times (V \otimes \mathbb{C}) \to \mathbb{C}$. We define $A : V \to V$ to be

orthogonal if $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ for all $\vec{v}$ and $\vec{w}$ in V. Our goal is to prove the following : If A is orthogonal then we can decompose V as $V_1 \oplus V_2 \oplus \cdots \oplus V_r$, with the V_i orthogonal, such that A carries each V to itself, wither by $\begin{bmatrix}\pm 1\end{bmatrix}$ or $\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$.

Problem 2.1. *Suppose that A has a real eigenvalue λ, with corresponding eigenvector $\vec{v}$. Show that $\lambda = \pm 1$.*

Proof. Since A is orthogonal, we have $\vec{v} \cdot \vec{v} = (A\vec{v}) \cdot (A\vec{v}) = \lambda\vec{v} \cdot \lambda\vec{v} = \lambda^2(\vec{v} \cdot \vec{v})$. Therefore $\lambda = \pm 1$. □

Problem 2.2. *Continue to assume that A has a real eigenvalue λ, with corresponding eigenvector $\vec{v}$. Show that $A : \vec{v}^\perp \to \vec{v}^\perp$, so we can induct.*

Proof. For any vector $\vec{w} \perp \vec{v}$, we have $0 = \vec{v} \cdot \vec{w} = (A\vec{v} \cdot A\vec{w}) = \lambda\vec{v} \cdot A\vec{w}$. Since $\lambda = \pm 1 \neq 0$, $\vec{v} \cdot A\vec{w} = 0$. Therefore $A : \vec{v}^\perp \to \vec{v}^\perp$. □

Suppose that $A : V \otimes \mathbb{C} \to V \otimes \mathbb{C}$ has an eigenvalue λ which is not in $\mathbb{R}$. Let the corresponding eigenvector be $\vec{v} = \vec{x} + i\vec{y}$, with $\vec{x}$ and $\vec{y} \in V$.

Problem 2.3. *Show that $\vec{x} \cdot \vec{x} = \vec{y} \cdot \vec{y}$ and $\vec{x} \cdot \vec{y} = 0$*

Proof. $(\vec{x}+i\vec{y})\cdot(\vec{x}+i\vec{y}) = A(\vec{x}+i\vec{y})\cdot A(\vec{x}+i\vec{y}) = \lambda^2(\vec{x}+i\vec{y})\cdot(\vec{x}+i\vec{y})$. Since $\lambda \notin \mathbb{R}$, $\lambda^2 \neq 1$ and $(\vec{x}+i\vec{y})\cdot(\vec{x}+i\vec{y}) = 0$. Therefore $(\vec{x}\cdot\vec{x} - \vec{y}\cdot\vec{y}) + 2i(\vec{x}\cdot\vec{y}) = 0 \Rightarrow \vec{x}\cdot\vec{x} = \vec{y}\cdot\vec{y}$ and $\vec{x} \cdot \vec{y} = 0$. □

Problem 2.4. *Show that $\lambda\bar{\lambda} = 1$, so $\lambda = e^{i\theta}$ for some real θ.*

Proof. Since $(\vec{x}+i\vec{y})\cdot(\vec{x}-i\vec{y}) = A(\vec{x}+i\vec{y})\cdot A(\vec{x}-i\vec{y}) = \lambda\bar{\lambda}(\vec{x}+i\vec{y})\cdot(\vec{x}-i\vec{y})$, either $\lambda\bar{\lambda} = 1$ or $(\vec{x}+i\vec{y})\cdot(\vec{x}-i\vec{y}) = 0$. However since $(\vec{x}+i\vec{y})\cdot(\vec{x}-i\vec{y}) = \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} \neq 0$, $\lambda\bar{\lambda} = 1$. Therefore $\lambda = e^{i\theta}$ for some real θ. □

Problem 2.5. *Put $L = Span(\vec{x}, \vec{y})$. Show that A preserves L and $L^\perp$, and acts on L by*

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

Proof. $e^{i\theta}(\vec{x} + i\vec{y}) = A(\vec{x} + i\vec{y}) = A\vec{x} + iA\vec{y}$ Sub $e^{i\theta} = \cos\theta + i\sin\theta$. $x\cos\theta - y\sin\theta + i(x\sin\theta + y\cos\theta) = A\vec{x} + iA\vec{y}$. Since A is real, it sends real vectors to real vectors. Thus we get $x\cos\theta - y\sin\theta = Ax$ and $x\sin\theta + y\cos\theta = Ay$. This concludes our proof. □

Problem 2.6. *Show that the exponential map $exp : \mathfrak{so}(n) \to SO(n) issurjective$*

Proof. For any $U \in SO(n)$, we can find a orthogonal matrix Q such that $QUQ^\intercal = D$, where D is a block-diagonal matrix whose blocks are $\begin{bmatrix}\pm 1\end{bmatrix}$ or $\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$. Since $exp(QMQ^\intercal) = Qexp(M)Q^\intercal$, it suffice to find a matrix M such that $exp(M) = D$. Since $exp(\begin{bmatrix}0\end{bmatrix}) = \begin{bmatrix}1\end{bmatrix}$, $exp(\begin{bmatrix}0 & -\theta \\ \theta & 0\end{bmatrix}) =$

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, and $exp(\begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and that the number of -1 on the diagonal of D is even given that the $D \in SO(n)$, we can first, by change of basis of D, put all the -1s on the diagonal together and take M to be the block diagonal matrix with $\begin{bmatrix} 0 \end{bmatrix}$ corresponding to $\begin{bmatrix} 1 \end{bmatrix}$ in D, $\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$ corresponding to $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ in D, and $\begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}$ corresponding to $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ in D and get $exp(M) = D$

□