Jordan Normal Form Day One

Let V be a finite dimensional vector space (over some field) and let $A : V \to V$ be a linear map. We recall that A is called ***invertible*** if A is bijective, and is called nilpotent if there is a positive integer n for which $A^n = 0$.

The goal of our first several problems is to show that V can be decomposed as a direct sum $V = I \oplus N$, where A maps each of I and N to themselves, where the restriction of A to I is invertible and the restriction of A to N is nilpotent.

Let I_j be the image of A^j, and let N_j be the kernel (also known as null space) of A^j.

Problem 1. Show that

$$I \supseteq I_2 \supseteq I_3 \supseteq \cdots \text{ and } N \subseteq N_2 \subseteq N_3 \subseteq \cdots .$$

Put $I = \bigcap_{j=1}^{\infty} I_j$ and $N = \bigcup_{j=1}^{\infty} N_j$.

Problem 2. Show that I and N are subspaces of V.

Problem 3. Show that A maps I to I and maps N to N.

Problem 4. Show that $A : I \to I$ is invertible and that $A : N \to N$ is nilpotent.

Jordan Normal Form Day Two

Let V be a finite dimensional vector space over a field k and $A : V \to V$ a linear map. Suppose that there are $\lambda_1, \lambda_2, \ldots, \lambda_r$ in k and positive integers $a_1, a_2, \ldots, a_r$ such that $\prod_{i=1}^{r}(A - \lambda_i \mathrm{Id})^{a_i} = 0$. Our goal for the next several problems is to prove that V decomposes as $V_1 \oplus V_2 \oplus \cdots \oplus V_r$ where A maps each V_i to itself and $A - \lambda_i \mathrm{Id}$ is nilpotent on V_i. Our proof is by induction on r.

Problem 5. Explain why the base case, $r = 1$, is trivial.

Problem 6. Now suppose that $r \geq 2$. From our previous results, we know that we can write $V = I \oplus N$ where $A - \lambda_r \mathrm{Id} : I \to I$ is invertible and $A - \lambda_r \mathrm{Id} : N \to N$ is nilpotent. Show that $\prod_{i=1}^{r-1}(A - \lambda_i \mathrm{Id})^{a_i} = 0$ on the subspace I. (Notice that the top index is $r - 1$ now!)

Problem 7. Finish the proof: Show that V decomposes as $V_1 \oplus V_2 \oplus \cdots \oplus V_r$ where A maps each V_i to itself and $A - \lambda_i$ is nilpotent on V_i.

Jordan Normal Form Day Three

Let W be a finite dimensional vector space over a field k and let $B : W \to W$ be nilpotent (meaning $B^m = 0$ for some m). Let $n = \dim W$. It this problem, we will show that we can find $j_1, j_2, \ldots, j_s$ with $j_1 + j_2 + \cdots + j_s = n$ and a basis e^p_q, with $1 \leq p \leq s$ and $1 \leq q \leq j_p$ such that B acts on this basis by

$$\begin{array}{l} 0 \leftarrow\!\shortmid e^1_1 \leftarrow\!\shortmid e^1_2 \leftarrow\!\shortmid e^1_3 \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid e^1_{j_1} \\ 0 \leftarrow\!\shortmid e^2_1 \leftarrow\!\shortmid e^2_2 \leftarrow\!\shortmid e^2_3 \leftarrow\!\shortmid e^2_4 \leftarrow\!\shortmid e^2_5 \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid e^2_{j_2} \\ \vdots \\ 0 \leftarrow\!\shortmid e^s_1 \leftarrow\!\shortmid e^s_2 \leftarrow\!\shortmid \cdots \leftarrow\!\shortmid e^s_{j_s} \end{array} \qquad (*)$$

Our proof is by induction on n. The base case $n = 0$ is trivial, so we assume $n > 0$. Let $\overline{W}$ be the image of W.

Problem 8. Show that $\dim \overline{W} < \dim W$.

Problem 9. Show that B maps $\overline{W}$ to itself.

By induction, we can find $\bar{j}_1, \bar{j}_2, \ldots, \bar{j}_{\bar{s}}$ and a basis e^p_q for $\overline{W}$ as above.

Problem 10. Show that, for each p, you can find a vector $e^p_{\bar{j}_p+1}$ in W such that $Be^p_{\bar{j}_p+1} = e^p_{\bar{j}_p}$.

So we now have vectors obeying $(*)$, but they aren't a basis yet.

Problem 11. Show that the vectors e^p_q which you have constructed so far are linearly independent.

Choose some additional vectors $f_1, f_2, \ldots, f_t$ such that the e^p_q you have already constructed, together with $f_1, f_2, \ldots, f_t$ form a basis for W.

Problem 12. Explain why there are constants c^p_q (dependent on r) such that

$$Bf_r = \sum_{p=1}^{s} \sum_{q=1}^{\bar{j}_p} c^p_q e^p_q.$$

Put

$$g_r = f_r - \sum_{p=1}^{s} \sum_{q=1}^{\bar{j}_p} c^p_q e^p_{q+1}.$$

Problem 13. Show that the e^p_q, together with g_r, is the desired basis.