Quotients of Lie groups

Let $H \subset G \subset \mathrm{GL}_n\mathbb{R}$ be Lie groups, with H closed in G. The aim of this problem is to show that G/H is a $\dim G - \dim H$ dimensional manifold.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H. Choose a vector subspace $\mathfrak{a}$ of $\mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$. Choose open neighborhoods U of 0 in $\mathfrak{g}$ and V of Id in G such that $\exp : U \to V$ and $\exp : (U \cap \mathfrak{h}) \to (V \cap H)$ are bijections with smooth inverses. Put $A = \exp(U \cap \mathfrak{a})$. So A is a manifold with $T_{\mathrm{Id}} A = \mathfrak{a}$.

Define

$$\mu : A \times H \to G$$

by $\mu(a, h) = ah$.

Problem 1. Show that there are open sets $A' \subset A$ and $W \subset H$, containing Id, such that $\mu : A' \times W \to \mu(A' \times W)$ is an injection with open image and smooth inverse.

Problem 2. Show that, for A'' any open subset of A', the set $\mu(A'' \times H)$ is open in G.

Our next goal is to show that we can take A'' a small enough neighborhood of Id that μ becomes injective on $A'' \times H$.

Problem 3. Suppose for the sake of contradiction that no such A'' exists. Show that there is a sequence of points $a_k \in A$ and $h_k \in H - \{\mathrm{Id}\}$ such that $\lim_{k\to\infty} a_k = \lim_{k\to\infty} h_k a_k = \mathrm{Id}$.

Problem 4. Show that, for all sufficiently large k, the above sequence h_k lies in the set W. Derive a contradiction to our assumption that $h_k \neq \mathrm{Id}$.

We now have an open set A'' such that $\mu(A'' \times H)$ is open in G and $\mu : A'' \times H \to \mu(A'' \times H)$ is bijective with smooth inverse.

Problem 5. Show that A'' and $\mu(A'' \times H)/H$ are homeomorphic.

Problem 6. Show that there is an open neighborhood of the coset H in G/H which is homeomorphic to A''. Show that every point of G/H has an open neighborhood homeomorphic to A''.