Compact Lie groups, abelian Lie groups

You may assume throughout the (true) fact that all the results you worked out for the matrix exponential with real matrices are also true with complex matrices.

Problem 1. Let A be an $n \times n$ complex matrix. Let $\exp(\mathbb{R}A)$ be the set of matrices of the form $\exp(tA)$ for $t \in \mathbb{R}$. Show that $\exp(tA)$ is bounded if and only if A is diagonalizable with purely imaginary eigenvalues. (This is where you should use Jordan normal form.)

Let $T \subset \mathrm{GL}_n(\mathbb{C})$ be a compact abelian Lie group, with Lie algebra $\mathfrak{t}$.

Problem 2. Show that any two matrices, A and B, in $\mathfrak{t}$, commute.

Problem 3. (This is a slight variant of a homework problem.) Let A_1, A_2, $\ldots$, A_r be commuting matrices, each of which is diagonalizable. Show that there is a change of basis which diagonalizes them all at once.

Problem 4. (Putting the parts together) Show that there is a change of basis, which puts $\mathfrak{t}$ inside the vector space of diagonal entries with purely imaginary entries.

Maximal tori

Let G be a **compact** subgroup of $\mathrm{GL}_n\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra. We define a ***torus*** T of G to be a compact connected abelian subgroup. We call T is a ***maximal torus*** of G if it is not contained in any torus of larger dimension.

Let T be a torus of G and let $\mathfrak{t}$ be its Lie algebra. Let $V = \{v \in \mathfrak{g} : [\theta, v] = 0 \text{ for all } \theta \in \mathfrak{t}\}$.

Problem 5. Show that $V \supseteq \mathfrak{t}$.

Problem 6. Suppose that $V = \mathfrak{t}$. Show that T is maximal.

Our goal now is to prove the converse of the previous problem: If $V \supsetneq \mathfrak{t}$ then T is not maximal. Let $v \in V$ with $v \notin \mathfrak{t}$. Let U be the vector space spanned by $\mathfrak{t}$ and v.

Problem 7. Show that there is some $s \in \mathrm{GL}_n\mathbb{C}$ such that $U \subseteq s\mathfrak{d}s^{-1}$. (Recall that G is compact.)

For the next problem, you'll want this Lemma. I suggest you think about it last.

Lemma: Let G and H be Lie subgroups of $\mathrm{GL}_n\mathbb{R}$ (or $\mathrm{GL}_n\mathbb{C}$) with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Then $G \cap H$ is a Lie subgroup with Lie algebra $\mathfrak{g} \cap \mathfrak{h}$.

Problem 8. Show that the connected component of the identity in $G \cap sDs^{-1}$ is a torus containing T, and of larger dimension.

If you have time left:

Problem 9. Prove the Lemma.

Finishing up maximal tori

Let G be a **compact** subgroup of $\mathrm{GL}_n\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra. We define a ***torus*** T of G to be a compact connected abelian subgroup. We call T is a ***maximal torus*** of G if it is not contained in any torus of larger dimension. We write D for the group $\{\mathrm{diag}(e^{i\theta_1}, \ldots e^{i\theta_n}) : \theta_1, \ldots, \theta_n \in \mathbb{R}\}$ and $\mathfrak{d}$ for its Lie algebra.

Let $T \subset G$ be a torus with lie algebra $\mathfrak{t}$. Put $V = \{v \in \mathfrak{g} : [\theta, v] = 0 \text{ for all } \theta \in \mathfrak{t}\}$. Our goal last time was to prove:

Theorem: T is maximal if and only if $V = \mathfrak{t}$.

Last time we showed:

- $\mathfrak{t} \subsetneq V$.
- If T is **not** maximal then $V \neq \mathfrak{t}$.

Let's finish up: Let $v \in V$ with $v \notin \mathfrak{t}$. Let U be the vector space spanned by $\mathfrak{t}$ and v.

Problem 10. Show that there is some $s \in \mathrm{GL}_n\mathbb{C}$ such that $U \subseteq s\mathfrak{d}s^{-1}$. (Recall that G is compact.)

Problem 11. Show that the connected component of the identity in $G \cap sDs^{-1}$ is a torus containing T, and of larger dimension.

Note that this proof shows that a maximal torus can be conjugated into D. However, we don't define a maximal torus to be a subgroup of D. The definition is the one given above.

Maximal tori: Examples

Problem 12. Show that D is a maximal torus of $U(n)$.

Problem 13. Find a maximal torus in $SU(n)$.

Problem 14. Show that

$$\left\{ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \;:\; \theta \in \mathbb{R} \right\}$$

is a maximal torus of $SO(3)$.

Problem 15. Find a maximal torus in $SO(n)$.