The unitary group

Let V be a complex vector space. A ***sesquilinear form*** is a map $* : V \times V \to \mathbb{C}$ obeying:

$$\begin{aligned}
\vec{u} * (\vec{v} + \vec{w}) &= \vec{u} * \vec{v} + \vec{u} * \vec{w} \\
(\vec{u} + \vec{v}) * \vec{w} &= \vec{u} * \vec{w} + \vec{v} * \vec{w} \\
\lambda(\vec{u} * \vec{v}) &= (\overline{\lambda}\vec{u}) * \vec{v} = \vec{u} * (\lambda\vec{v})
\end{aligned}$$

A sesquilinear form is called ***Hermitian*** if it obeys

$$\vec{u} * \vec{v} = \overline{\vec{v} * \vec{u}}.$$

Note that this implies that $\vec{v} * \vec{v} \in \mathbb{R}$. A Hermitian bilinear form is called ***positive definite*** if, for all nonzero $\vec{v}$, we have $\vec{v} * \vec{v} = 0$.

If we identify V with $\mathbb{C}^n$, then sequilinear forms are of the form $\vec{v} * \vec{w} = \vec{v}^\dagger Q \vec{w}$ for an arbitrary $Q \in \mathrm{Mat}_{n \times n}(\mathbb{C})$, the Hermitian condition says that $Q = Q^\dagger$, and the positive definite condition says that Q is positive definite Hermitian. The standard Hermitian form on $\mathbb{C}^n$ is $\vec{v}^\dagger \vec{w}$.

Let V be a finite complex vector space equipped with a positive definite Hermitian form $*$. We define $A : V \to V$ to be ***unitary*** if $(A\vec{v}) * (A\vec{w}) = \vec{v} * \vec{w}$ for all $\vec{v}$ and $\vec{w}$ in V. We will show that V has a $*$-orthonormal basis of eigenvectors, whose eigenvalues are all of the form $e^{i\theta}$. Our proof is by induction on n.

Let λ be a complex eigenvalue of A, with eigenvector $\vec{v}$.

Problem 1. Show that $|\lambda| = 1$.

Problem 2. Show that A takes $\{\vec{w} : \vec{v} * \vec{w} = 0\}$ to itself.

Problem 3. Explain why we are done.

The orthogonal group

On this page we make sure to get a clear record of a result which was done confusingly in homework: A real orthogonal matrix can be put into block diagonal form where the blocks are $\left[\begin{smallmatrix} \pm 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \cos\theta & -\sin\theta \end{smallmatrix}\right]$.

Let V be a finite dimensional real vector space equipped with a positive definite symmetric bilinear form $\cdot : V \times V \to \mathbb{R}$. We can then extend $\cdot$ to a symmetric bilinear form $\cdot : (V \otimes \mathbb{C}) \times (V \otimes \mathbb{C}) \to \mathbb{C}$. We define $A : V \to V$ to be ***orthogonal*** if $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ for all $\vec{v}$ and $\vec{w}$ in V. Our goal is to prove the following: If A is orthogonal then we can decompose V as $V_1 \oplus V_2 \oplus \cdots \oplus V_r$, with the V_i orthogonal, such that A carries each V to itself, either by $\left[\begin{smallmatrix} \pm 1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$.

Problem 4. Suppose that A has a real eigenvalue λ, with corresponding eigenvector $\vec{v}$. Show that $\lambda = \pm 1$.

Problem 5. Continue to assume that A has a real eigenvalue λ, with corresponding eigenvector $\vec{v}$. Show that $A : \vec{v}^{\perp} \to \vec{v}^{\perp}$, so we can induct.

Suppose that $A : V \otimes \mathbb{C} \to V \otimes \mathbb{C}$ has an eigenvalue λ which is not in $\mathbb{R}$. Let the corresponding eigenvector be $\vec{v} = \vec{x} + i\vec{y}$, with $\vec{x}$ and $\vec{y} \in V$.

Problem 6. Show that $\vec{x} \cdot \vec{x} = \vec{y} \cdot \vec{y}$ and $\vec{x} \cdot \vec{y} = 0$.

Problem 7. Show that $\lambda\overline{\lambda} = 1$, so $\lambda = e^{i\theta}$ for some real θ.

Problem 8. Put $L = \mathrm{Span}(\vec{x}, \vec{y})$. Show that A preserves L and $L^{\perp}$, and acts on L by $\left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$.

Let's do something which isn't on the homework!

Problem 9. Show that the exponential map $\exp : \mathfrak{so}(n) \to \mathrm{SO}(n)$ is surjective.