Problem Set 1 – due Friday January 12

See the course website for policy on collaboration.

1. Let V be a vector space and W a sub-vector space. This problem reviews the definition of the quotient vector space, V/W, which I hope many of you have already seen. Define an equivalence relation $\sim$ on V by $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$.
   (a) Let $X_1 \subset V$ and $X_2 \subset V$ be equivalence classes for $\sim$. Define $X_1 + X_2 = \{x_1 + x_2 : \ x_1 \in X_1, \ x_2 \in X_2\}$. Show that $X_1 + X_2$ is a single equivalence class for $\sim$.
   (b) Let X be an equivalence class for $\sim$ and a a scalar. Define $aX = \{ax : \ x \in X\}$. Show that aX is an equivalence class for $\sim$.
   (c) Show that the operations of addition and scalar multiplication defined in parts (a) and (b) make the set of equivalence classes for $\sim$ into a vector space. This vector space is called V/W.

2. Let V be a real vector space of dimension at least 2. For each of the following supposed constructions, state whether or not this construction is well defined and prove your answer.
   (a) Define a linear map $V \to V \otimes V$ by $v \mapsto v \otimes v$ for all $v \in V$.
   (b) Define a linear map $V \otimes V \to V$ by $v \otimes w \mapsto v + w$.
   (c) Let $\langle \ , \ \rangle$ be a symmetric bilinear form on V. Define a symmetric bilinear form $(\ , \ )$ on $V \otimes V$ by $(u \otimes v, \ w \otimes x) = \langle u, w \rangle \langle v, x \rangle$.
   (d) Let $\langle \ , \ \rangle$ be a symmetric bilinear form on V. Define a symmetric bilinear form $(\ , \ )$ on $V \otimes V$ by $(u \otimes v, \ w \otimes x) = \langle u, v \rangle \langle w, x \rangle$.

3. Let V be a real vector space with basis $e_1, e_2, \ldots, e_n$. Let $\alpha = \sum_{i,j=1}^{n} a_{ij} e_i \otimes e_j$ be an element of $V \otimes V$. Show that α is of the form $v \otimes w$ if and only if the matrix a_{ij} has rank ≤ 1.

4. Let V be a real vector space of dimension n. We consider $\mathbb{C}$ as a two dimensional real vector space, so $\mathbb{C} \otimes V$ is a real vector space of dimension $2n$. For $\beta \in \mathbb{C}$ and $\phi = \sum \gamma_i \otimes v_i \in \mathbb{C} \otimes V$, we define $\beta\phi = \sum (\beta\gamma_i) \otimes v_i$.
   (a) Show that $\beta\phi$ is a well defined element of $\mathbb{C} \otimes V$.
   (b) With this definition of scalar multiplication (and the same definition of addition as before), show that $\mathbb{C} \otimes V$ becomes a complex vector space.
   (c) If $e_1, \ldots, e_n$ is a basis for V as a real vector space, show that $1 \otimes e_1, 1 \otimes e_2, \ldots, 1 \otimes e_n$ is a basis of $\mathbb{C} \otimes V$ as a complex vector space.

5. This problem build on problem 5 from the final exam. Let $G \subset \mathrm{GL}_n(\mathbb{R})$ be a 3-dimensional Lie group with Lie Algebra $\mathfrak{g}$. Suppose that we can identify $\mathfrak{g}$ with $\mathbb{R}^3$ such that $[X, Y] = (X \times Y)$, where $\times$ is the ordinary cross product.

   Let S be the sphere of radius 2π in $\mathbb{R}^3$. In that problem, you showed that exp is constant on S; put $\exp(S) = z$.
   (a) Show that z commutes with $\exp(W)$, for any $W \in \mathfrak{g}$. (This part is repeated from the exam.)
   (b) Show that $z^2 = \mathrm{Id}$.
   Let $H = \{g \in G : gz = zg\}$.
   (c) Show that H is a subgroup of G.
   (d) Show that H is closed in G.
   (e) Show that H is open in G.