Problem Set 10 – due Friday April 6

See the course website for policy on collaboration.

1. Let G be a Lie group, meaning a smooth manifold with a smooth multiplication map $G \times G \to G$ making G into a group with identity e. Let $\mathfrak{g} = T_e G$. We write C_g for the map $h \mapsto ghg^{-1}$ from G to G. Put $\chi(g) = \det \Big((DC_g)_e : \mathfrak{g} \to \mathfrak{g} \Big)$, so $\chi(g) \in \mathbb{R}_{\neq 0}$.
   (a) Show that $\chi(g_1)\chi(g_2) = \chi(g_1 g_2)$.
   (b) Compute the function χ on the group of matrices of the form $\left[\begin{smallmatrix} u & v \\ 0 & 1 \end{smallmatrix}\right]$, with $u \neq 0$.
   (c) If G is compact and connected, show that $\chi(g) = 1$. (Hint: Consider the maximum value of $\chi(g)$.)

2. Let T be the two dimensional torus $S^1 \times S^1$. We'll define R to be the rectangle $[0, 2\pi] \times [0, 2\pi]$ and $f : R \to T$ to be the map $(x, y) \mapsto ((\cos x, \sin x), (\cos y, \sin y))$. In other words, f glues the sides of the rectangle together. For $y \in [0, 2\pi]$, let γ_y be the closed curve $\{f(x, y) : x \in [0, 2\pi]\}$ in T and let δ_x be the closed curve $\{f(x, y) : y \in [0, 2\pi]\}$.

   Let ω and η be **closed** 1-forms on T. In this problem, we prove the relation:
$$\int_T \omega \wedge \eta = \det \begin{bmatrix} \int_\gamma \omega & \int_\gamma \eta \\ \int_\delta \omega & \int_\delta \eta \end{bmatrix}.$$
   To be specific, we should nail down orientations: We choose orientations such that $dx \wedge dy$ is positive on T, that dx is positive on γ and dy is positive on δ.
   (a) Show that $\int_{\gamma_y} \omega$ does not depend on y, and likewise for the other three one dimensional integrals above.
   We write $\int_\gamma \omega$ for the value which $\int_{\gamma_y} \omega$ takes for all y. This explains the meaning of the right hand side of the above equation.
   By Poincare's lemma, $f^*\omega$ is exact on R; put $f^*\omega = dp$.
   (b) Show that $p(2\pi, y) - p(0, y) = \int_\gamma \omega$ and $p(x, 2\pi) - p(x, 0) = \int_\delta \omega$
   (c) Prove the above relationship between $\int_T \omega \wedge \eta$ and the four one dimensional integrals $\int_\gamma \omega$, $\int_\gamma \eta$, $\int_\delta \omega$ and $\int_\delta \eta$.

3. In this problem, we study the orientability of $\mathbb{RP}^{n-1}$. We define $\mathbb{RP}^{n-1}$ as $(\mathbb{R}^n \setminus \{0\}) / \sim$, where two vectors are equivalent if they are proportional. We write $x_1, x_2, \ldots, x_n$ be the coordinates on $\mathbb{R}^n$. Let $\tilde{U}_j$ be the open set $x_j \neq 0$ in $\mathbb{R}^n$, and let U_j be its image in $\mathbb{RP}^{n-1}$. We will take P_j to be isomorphic to $\mathbb{R}^{n-1}$, with coordinates called $(x_{1,j}, x_{2,j}, \ldots, x_{(j-1),j}, x_{(j+1),j}, \ldots, x_{n,j})$, and our map $f_j : P_j \to \mathbb{RP}^{n-1}$ is given by sending $(x_{1,j}, x_{2,j}, \ldots, x_{(j-1),j}, x_{(j+1),j}, \ldots, x_{n,j})$ to the equivalence class of $(x_{1,j}, x_{2,j}, \ldots, x_{(j-1),j}, 1, x_{(j+1),j}, \ldots, x_{n,j})$.
   (a) Show that $f_k^{-1}(U_j \cap U_k)$ is the open set $x_{j,k} \neq 0$, and give formulas for the change of coordinates between P_j and P_k.
   (b) If n is even (so $\mathbb{RP}^{n-1}$ is odd dimensional), show that $(-1)^{j-k} \det D(f_j^{-1} \circ f_k) > 0$ at every point where it is defined. Deduce (from statements in class) that $\mathbb{RP}^{n-1}$ is orientable.
   (c) If n is odd, show that $\mathbb{RP}^{n-1}$ is not orientable.

On the back of the page, we justify the statements about orientations made in class and used in the previous problem.

4. This question fills in the statements about orientations made in class. Let X be a smooth, Hausdorff, d-dimensional manifold.

   We will define a ***patch path*** to be a sequence of points x_0, x_1, …, x_N in X and a sequence of patches (f_1, P_1, U_1), (f_2, P_2, U_2), …, (f_N, P_N, U_N) such that x_{k-1} and x_k are in U_k and $f_k^{-1}(x_{k-1})$ and $f_k^{-1}(x_k)$ lie in the same connected component of P_k. (Note: "patch path" is not a standard term.)

   (a) Suppose that X is oriented. Let (f_1, P_1, U_1), …, (f_N, P_N, U_N) and x_0, x_1, …, x_N be a patch path with $x_0 = x_N$. For notational convenience, we put $(f_{N+1}, P_{N+1}, U_{N+1}) = (f_1, P_1, U_1)$.
   Show that
   $$\prod_{k=1}^{N} \det D\left(f_{k+1}^{-1} \circ f_k\right)_{f_k^{-1}(x_k)} > 0 \qquad (*).$$

   (b) For y and $z \in X$, define $y \sim z$ if there is a patch path with $y = x_0$ and $z = x_N$. Show that $\sim$ is an equivalence relation.

   (c) Show that the equivalence classes of $\sim$ are both closed and open. Deduce that, if X is connected, then there is a patch path from any point of X to any other point.

   The rest of the problem establishes the converse of part (a). Fix an atlas on X. We'll say that a patch path is in our atlas if each (f_k, P_k, U_k) is in our atlas.

   From now on, assume that, for every patch path in our atlas, the left hand side of $(*)$ is positive. We will build an orientation on X.

   (d) Let $(f_j, P_j, U_j)_{1 \le j \le M}$ and $(x_j)_{0 \le j \le M}$ be a patch path, and $(g_k, Q_k, V_k)_{1 \le k \le N}$ and $(y_k)_{0 \le k \le N}$ be another, with $(f_1, P_1, U_1) = (g_1, Q_1, V_1)$, $(f_M, P_M, U_M) = (g_N, Q_N, V_N)$, $x_0 = y_0$ and $x_M = y_N$. Show that
   $$\prod_{j=1}^{M-1} \det D\left(f_{j+1}^{-1} \circ f_j\right)_{f_j^{-1}(x_j)} \text{ has the same sign as } \prod_{k=1}^{N-1} \det D\left(g_{k+1}^{-1} \circ g_k\right)_{g_k^{-1}(y_k)}.$$

   For notational simplicity, assume X is connected. Choose a point $y \in X$ and a patch (f_0, P_0, U_0) around y in our atlas. For any other point $z \in X$, choose a patch path $(f_j, P_j, U_j)_{1 \le j \le M}$ whose first patch is (f_0, P_0, U_0), with $x_0 = y$ and $x_M = z$. Define an orientation on X by choosing the ray
   $$\prod_{j=1}^{M-1} \det D\left(f_{j+1}^{-1} \circ f_j\right)_{f_j^{-1}(x_j)} \bigwedge^d D(f_M)_{f_M^{-1}(z)} \left(\mathbb{R}_{>0} e_1 \wedge e_2 \wedge \cdots \wedge e_d\right)$$
   as the positive component of $\bigwedge^d T_z X \setminus \{0\}$. Here e_1, …, e_d is the standard basis of $\mathbb{R}^d$.

   (e) Show that this component of $\bigwedge^d T_z X \setminus \{0\}$ does not depend on the choice of patch path from y to z.