Problem Set 11 – due Friday April 13

See the course website for policy on collaboration.

1. This question constructs what is known as the Haar measure on a Lie group. Let G be a n-dimensional Lie group with Lie algebra $\mathfrak{g}$. For $g \in G$, let L_g be the map $h \mapsto gh$ from G to G and let R_g be the map $h \mapsto hg$. Fix a nonzero element $\omega \in \bigwedge^n \mathfrak{g}^\vee$. Define an n-form μ^L on G by
$$\mu^L_g(\vec{v}_1, \ldots, \vec{v}_n) = \omega(DL_{g^{-1}}\vec{v}_1, \ldots, DL_{g^{-1}}\vec{v}_n).$$
The form μ^L is called the ***left Haar measure***. We similarly define right Haar measure by
$$\mu^R_g(\vec{v}_1, \ldots, \vec{v}_n) = \omega(DR_{g^{-1}}\vec{v}_1, \ldots, DR_{g^{-1}}\vec{v}_n).$$
   (a) For any $h \in G$, show that $L_h^* \mu^L = \mu^L$ and $R_h^* \mu^R = \mu^R$.

   (b) Let $\chi : G \to \mathbb{R}$ be the function from the previous problem set. Show that $\mu^R_g = \chi(g)\mu^L_g$. Deduce that, if G is a compact connected Lie group, then $\mu^L = \mu^R$.

   As an application, we prove the following theorem: If G is a compact Lie group, contained in some GL_N, then G preserves a positive definite inner product on $\mathbb{R}^N$.

   Choose the orientation $\mathcal{O}$ of G such that μ_g is positive on the component $\mathcal{O}$ of $\bigwedge^n T_g G$. Let $\cdot$ be the standard dot product on $\mathbb{R}^N$. Define
$$\vec{u} * \vec{v} = \int_{g \in G} (g\vec{u}) \cdot (g\vec{v}) \ \mu^R.$$
   (c) Show that $(g\vec{u}) * (g\vec{v}) = \vec{u} * \vec{v}$ for all $g \in G$.

   (d) Show that $\vec{v} * \vec{v} > 0$ for all nonzero $\vec{v}$.

2. A question about second countability and paracompactness! Let T be a topological space. Recall that a ***basis of neighborhoods*** for X is a collection $\{U_i\}$ of open sets such that, for any nonempty open set $V \subseteq X$ and any $x \in V$, there is a U_i with $x \in U_i \subseteq V$. The space T is called ***second countable*** if it has a countable basis of neighborhoods.

   (a) Show that $\mathbb{R}^n$ is second countable. Show that any subset of $\mathbb{R}^n$, with the subspace topology, is second countable. (Hint: There are only countably many rational numbers.)

   (b) Let X be a manifold with a countable atlas. Show that X is second countable.

   Let X be a k-dimensional manifold (including the condition that X is Hausdorff) with a countable atlas. We are going to show X is paracompact.

   (c) Show that we can equip X with a new (equivalent) atlas $\{(f_j, P_j, f_j(P_j))\}_{1 \leq i < \infty}$ where each P_j is an open ball of some radius r_j around some point $z_j \in \mathbb{R}^k$. We abbreviate the closed and open balls around z_j of radius r $B_j(r)$ and $\overline{B}_j(r)$.

   Let V_α be an open cover of X. We need to find a locally finite refinement. For $k \geq 1$, put
$$U_k = \bigcup_{j=1}^{k} f_j\left(B_j\Big(\frac{k-1}{k} r_j\Big)\right) \qquad K_k = \bigcup_{j=1}^{k} f_j\left(\overline{B}_j\Big(\frac{k-1}{k} r_j\Big)\right).$$
   For convenience, we define $U_k = K_k = \emptyset$ for $k \leq 0$.

   (d) Show that U_k and K_k form an exhaustion of X, meaning that $X = \bigcup U_k$, the U_k are open, the K_k are compact, and $U_1 \subset K_1 \subset U_2 \subset K_2 \subset \cdots$.

(e) Show that, for each k, we can find finitely many open sets $W_{k,1}$, $W_{k,2}$, $\ldots$, W_{k,M_k} such that each W_j is contained in some V_α and

$$K_k \setminus U_{k-1} \subseteq \bigcup_{\ell=1}^{M_k} W_{k,\ell} \subseteq U_{k+1} \setminus K_{k-2}.$$

(f) Explain why the set of all $W_{k,\ell}$ form a locally finite refinement of V_α.