Problem Set 2 – due Friday January 19

See the course website for policy on collaboration.

1. Let V be a real vector space of dimension n, with basis $e_1, e_2, \ldots, e_n$. Let $\omega \in \bigwedge^2 V$ and write $\omega = \sum p_{ij} e_i \otimes e_j$, so $p_{ij} = -p_{ji}$. Let P be the $n \times n$ skew symmetric matrix whose entries are the p_{ij}. In this problem, we will show that ω is of the form $u \wedge v$, for u and $v \in V$, if and only if P has rank ≤ 2.
   (a) Show that, if $\omega = u \wedge v$, then the matrix P has rank ≤ 2.
   (b) Show that, if P has rank ≤ 2, then we can write $\omega = u \wedge v$. (Hint for one approach: If P is zero, we are done. Otherwise, we may assume (why?) that $p_{12} = -p_{21} \neq 0$. Give explicit formulas for a choice of u and v of the form $u = e_1 + u_3 e_3 + u_4 e_4 + \cdots + u_n e_n$ and $v = e_2 + v_3 e_3 + v_4 e_4 + \cdots + v_n e_n$.)
   (c) True or false: If $\dim V = 3$, then every vector in $\bigwedge^2 V$ is of the form $u \wedge v$. What about for $\dim V = 4$?

2. Let V be a finite dimensional vector space. Let $A : V \to V$ be a linear map and suppose that A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
   (a) Describe the eigenvalues of the maps $A \otimes A$, $\mathrm{Sym}^2 A$ and $\bigwedge^2 A$, which respectively go from $V \otimes V$ to $V \otimes V$, from $\mathrm{Sym}^2 V$ to $\mathrm{Sym}^2 V$ and from $\bigwedge^2 V$ to $\bigwedge^2 V$.
   (b) Give formulas for $\mathrm{Tr}(A \otimes A)$, $\mathrm{Tr}(\mathrm{Sym}^2 A)$ and $\mathrm{Tr}(\bigwedge^2 A)$ in terms of $\mathrm{Tr}(A)$ and $\mathrm{Tr}(A^2)$.

3. Let A be an $m \times n$ matrix, which we also consider as a linear map $\mathbb{R}^n \to \mathbb{R}^m$. Let k be a positive integer. Recall that the rank of A is **defined** to be the dimension of the image of A. On the zeroeth problem set of Fall term, where we showed that $\mathrm{rank}(A) \geq k$ if and only if A has a $k \times k$ submatrix with nonzero determinant. In this problem, we give a slicker proof.
   (a) Show that A has a $k \times k$ matrix with nonzero determinant if and only if $\bigwedge^k A$ is nonzero.
   (b) Suppose that $\mathrm{rank}(A) < k$. Show that $\bigwedge^k A = 0$.
   (c) Suppose that $\mathrm{rank}(A) \geq k$. Show that $\bigwedge^k A \neq 0$. (Hint: Choose convenient bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ for this computation.)

4. For A an $m \times n$ matrix of complex numbers, we define $A^\dagger$ to be be the $n \times m$ matrix with $A_{ij} = \overline{A}_{ji}$. (Here $\overline{z}$ is the complex conjugate of z.) We define $U(n) \subset \mathrm{GL}_n\mathbb{C}$ to be the $n \times n$ matrices A for which $AA^\dagger = \mathrm{Id}_n$
   (a) Show that $(AB)^\dagger = B^\dagger A^\dagger$.
   (b) Show that $U(n)$ is a subgroup of $GL_n\mathbb{C}$.
   The next part of this question is modified. If you've already written up the old question, feel free to submit it. But, based on conversations in office hours, most of you would find this clearer: Let $\mathrm{Her}_n = \{X \in \mathrm{Mat}_{n\times n}\mathbb{C} : X = X^\dagger\}$; the elements of Her_n are called Hermitian matrices. Let $\mathfrak{u}(n) = \{X \in \mathrm{Mat}_{n\times n}\mathbb{C} : X + X^\dagger = 0\}$.
   (c) Show that Her_n is a real vector space of dimension n^2.
   Define $g : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Her}_n$ by $g(X) = XX^\dagger$.
   (d) Show that the linear map Dg_{Id_n} is surjective, and deduce that $U(n)$ is a Lie group.
   (e) Show that $\mathfrak{u}(n)$ is the Lie algebra of $U(n)$.
   Old version: In the interests of time, I'll let you assume that $U(n)$ is a manifold.
   (c) Show that the Lie algebra, $\mathfrak{u}(n)$ of $U(n)$, is $\{X \in \mathrm{Mat}_{n\times n}\mathbb{C} : X + X^\dagger = 0\}$.

5. This question assumes you already know the spectral theorem: If A is a symmetric real matrix, then there is an orthogonal matrix Q such that QAQ^T is diagonal. Show that, if A and B are symmetric real $n \times n$ matrices with $AB = BA$, then there is an orthogonal matrix such that both QAQ^T and QBQ^T are diagonal. (Hint: As a warm up, you might want to start with the case where the eigenvalues of A are distinct.)