PROBLEM SET 3 – DUE FRIDAY JANUARY 26

See the course website for policy on collaboration.

1. Consider the nilpotent matrix X whose powers and ranks are shown below:

$$X = \begin{bmatrix} -1 & -7 & -2 & -2 \\ 0 & 3 & 1 & 1 \\ 0 & -6 & -2 & -2 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad X^2 = \begin{bmatrix} -1 & -4 & -1 & -1 \\ 1 & 4 & 1 & 1 \\ -2 & -8 & -2 & -2 \\ -1 & -4 & -1 & -1 \end{bmatrix} \quad X^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{rank}(X) = 2 \qquad \mathrm{rank}(X^2) = 1 \qquad \mathrm{rank}(X^3) = 0$$

Give a basis for $\mathbb{R}^4$ which puts X into Jordan normal form.

2. Feel free to use technology to compute the eigenvalues and eigenvectors in this problem. Let $Z = \left[\begin{smallmatrix} 23 & 13 \\ -34 & -19 \end{smallmatrix}\right]$. Show how to factor Z in the form

$$Z = S \left[\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right] S^{-1}$$

for S a real 2×2 matrix and x and y real numbers.

3. Give an example of a symmetric **complex** matrix which is nilpotent but not zero.

4. Let U be an $n \times n$ real orthogonal matrix. Recall that this means $(U\vec{v}) \cdot (U\vec{w}) = \vec{v} \cdot \vec{w}$ for any vectors $\vec{v}$ and $\vec{w}$.
   (a) If λ is a real eigenvalue of U, show that $\lambda = \pm 1$.
   (b) Let λ be a complex eigenvalue of U, which is not real. Show that $\lambda = e^{i\theta}$ for some real number θ. (Hint: Let $\vec{v} \in \mathbb{C}^n$ be a λ-eigenvector, and consider the complex conjugate $\overline{\vec{v}}$.)
   (c) Let θ_1 and θ_2 be two angles with $e^{i\theta_1} \neq e^{\pm i\theta_2}$. Let $\vec{x}_j + i\vec{y}_j$ be an $e^{i\theta_j}$ eigenvector of U, with $\vec{x}_j$ and $\vec{y}_j$ in $\mathbb{R}^n$. Show that $\mathrm{Span}(\vec{x}_1, \vec{y}_1)$ and $\mathrm{Span}(\vec{x}_2, \vec{y}_2)$ are perpindicular.
   (d) Show that, for any orthogonal matrix V, we have $\mathrm{Ker}(V - \mathrm{Id})^2 = \mathrm{Ker}\, V - \mathrm{Id}$. (Hint: If $\vec{v} \in \mathrm{Ker}(V - \mathrm{Id})^2$ but not $\mathrm{Ker}\, V - \mathrm{Id}$, what does $V^n\vec{v}$ look like?)
   (e) Show that, for any real orthogonal matrix U, we can choose an orthonormal basis of $\mathbb{R}^n$ on which U is block diagonal, with blocks of the form $\left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$, $[\,1\,]$ and $[\,-1\,]$.

5. Let $N(k)$ be the $k \times k$ matrix where $N(k)_{ij}$ is 1 if $j = i + 1$ and 0 otherwise. For example,

$$N(4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $\vec{k} = (k_1, k_2, \ldots, k_r)$ a vector of positive integers with $k_1 \geq k_2 \geq \cdots \geq k_r$, let $N(\vec{k})$ be the block-diagonal matrix whose diagonal blocks are $N(k_1)$, $N(k_2)$, …, $N(k_r)$.
   (a) Give a formula for $\dim \mathrm{Ker}\, N(\vec{k})^j$ in terms of $\vec{k}$ and j.
   (b) Let $X : V \to V$ be a nilpotent matrix. We showed in class that X is similar to some $N(\vec{k})$. Show that X is similar to only on $N(\vec{k})$.
   Let $n = \sum k_i$ and let $U(\vec{k}) \subset \mathrm{Mat}_{n\times n}(\mathbb{R})$ be the set of matrices similar to $N(\vec{k})$. So the set of nilpotent $n \times n$ matrices decomposes into pieces, $U(\vec{k})$. For your convenience, we list the possible $\vec{k}$'s for $n = 4$:

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1) \qquad (*).$$

   (c) Let $\vec{k}$ and $\vec{\ell}$ be vectors in the list $(*)$. Determine, for which pairs $\vec{k}$ and $\vec{\ell}$, the space $U(\vec{k})$ is contained in the closure $\overline{U(\vec{\ell})}$.