Problem Set 4 – due Friday February 2

See the course website for policy on collaboration.

1. We review basic computations with 1-forms.
   (a) Let x and y be coordinates on $\mathbb{R}^2$. Write $d\,(e^x \sin y)$ as a linear combination of dx and dy.
   (b) Let u, v be coordinates on $\mathbb{R}^2$ and let x, y, z be coordinates on $\mathbb{R}^3$. Let $\phi(u,v) = (u\cos v, u\sin v, v)$. Compute $\phi^* d(x^2+y^2)$.

2. In this problem we will prove the following result: Let U be an open subset of $\mathbb{R}^n$ and p a point of U. Suppose there is a smooth map $h : U \times [0,1] \to U$ such that, for all $x \in U$, we have $h(x,1) = x$ and $h(x,0) = p$. Then any closed 1-form on U is exact.

   We'll write $(x_1, x_2, \ldots, x_n)$ for the coordinates on U and t for the coordinate on $[0,1]$. This will give us many more situations where we can say that closed is the same as exact.

   (a) Let $\omega = \sum f_i(x_1, \ldots, x_n, t) dx_i + g(x_1, \ldots, x_n, t) dt$ be a closed 1-form on $U \times [0,1]$, meaning that $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ and $\frac{\partial f_i}{\partial t} = \frac{\partial g}{\partial x_i}$. Put $G(x_1, \ldots, x_n) = \int_{t=0}^1 g(x_1, \ldots, x_n, t) dt$. Let ω_0 and ω_1 be the 1-forms on U obtained by restricting ω to $U \times \{0\}$ and $U \times \{1\}$. Show that
   $$dG = \omega_1 - \omega_0.$$
   (b) Let $h : U \times [0,1] \to U$ be as in the introduction to the problem. Let η be a closed 1-form on U. Show that $h^*\eta$ is closed.
   (c) Let $h : U \times [0,1] \to U$ be as in the introduction to the problem and let η be a closed 1-form on U. Combine the previous parts to show that η is exact, in other words, there is some function G on U with $dG = \eta$.
   (d) Suppose that, U is convex and nonempty, meaning that, for any points p and q in U, the line segment between p and q is in U. Show that there is a map $h : U \times [0,1] \to U$ as in the problem statement.

3. This question applies Stokes' Theorem to prove the Fundamental Theorem of Algebra.
   (a) Let $0 < p < q$ and let $A = \{(x,y) : p^2 \le x^2 + y^2 \le q^2\} \subset \mathbb{R}^2$. Let $C_1 = \{(x,y) : x^2 + y^2 = p^2\}$ and $C_2 = \{(x,y) : x^2 + y^2 = q^2\}$. Let $\phi : A \to \mathbb{R}^2 \setminus \{(0,0)\}$ be a smooth map. Show that
   $$\int_{C_1} \phi^* \left(\frac{x dy - y dx}{x^2 + y^2} \right) = \int_{C_2} \phi^* \left(\frac{x dy - y dx}{x^2 + y^2} \right).$$
   (b) Let n be a positive integer. Show that there is no smooth map $\phi : A \to \mathbb{R}^2 \setminus \{(0,0)\}$ such that $\phi(p\cos\theta, p\sin\theta) = (1,0)$ and $\phi(q\cos\theta, q\sin\theta) = (\cos(n\theta), \sin(n\theta))$. (Hint: Your computations will be easier if you recall that $\frac{xdy - ydx}{x^2+y^2} = d\tan^{-1}(y/x)$ whenever $x \neq 0$.)

   Let's prove the Fundamental Theorem of Algebra! Suppose for the sake of contradiction that $f(z) = a_n z^n + \cdots + a_0$ is a complex polynomial, with $a_n \neq 0$ and no complex zero. Let $F : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$ be
   $$F(x,y) = \frac{\text{Real}(f(x+iy)),\ \text{Im}(f(x+iy))}{(1 + \sqrt{x^2+y^2})^n}.$$
   Choose $0 < p < q$ and pick a continuous increasing function $h : (p,q) \to \mathbb{R}_{>0}$ with $\lim_{r \to p^+} h(r) = 0$ and $\lim_{r \to q^-} h(r) = \infty$. Put
   $$\phi(x,y) = F(h(\sqrt{u^2+v^2})u,\ h(\sqrt{u^2+v^2})v).$$
   (c) Show that ϕ, defined as above, extends to a continuous function $A \to \mathbb{R}^2 \setminus 0$.
   (d) Derive a contradiction. You may gloss over the issue of the smoothness of A at the boundary.

4. This question looks at the frequent issue of how to think about double derivatives abstractly. Let V be a finite dimensional real vector space, and $g : V \to \mathbb{R}$ a smooth function. Then $y \mapsto dg_y$ is a function $V \to V^*$. We'll denote this function as Dg. We can then take D of Dg so that, at every point $y \in V$, we have a linear map $D(Dg)_y : V \to V^*$. For $y \in V$ and $\vec{v}_1$, $\vec{v}_2$ in V, we write $(D^2 g)_y(\vec{v}_1, \vec{v}_2)$ to mean $D(Dg)_y(\vec{v}_1)(\vec{v}_2)$.

   (a) Show that $(D^2 g)_y(\vec{v}_1, \vec{v}_2) = (D^2 g)_y(\vec{v}_2, \vec{v}_1)$.

   Let U be another finite dimensional real vector space, and let $\phi : U \to V$ be a smooth map. Let $x \in U$ and $\vec{u}_1$, $\vec{u}_2 \in U$. We define $y = \phi(x)$ and $\vec{v}_1 = (D\phi)_x \vec{u}_1$, $\vec{v}_2 = (D\phi)_x \vec{u}_2$ and $f = g \circ \phi$.

   (b) If ϕ is linear, show that $(D^2 f)_x(\vec{u}_1, \vec{u}_2) = (D^2 g)_y(\vec{v}_1, \vec{v}_2)$.

   (c) Give an example to show that the previous result need not hold if ϕ is not linear. Hint: Take $\dim U = \dim V = 1$ and try almost anything.