Problem Set 5 – due Friday February 9

See the course website for policy on collaboration.

1. Define the 2-form
$$\omega = \frac{x dy\wedge dz - y dx\wedge dz + z dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$$
on $\mathbb{R}^3 \setminus \{0\}$.
   (a) Compute $d\omega$
   (b) Let $\sigma(r,\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$. Compute $\sigma^*\omega$.

2. Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be a smooth function. We consider $n-1$ forms on $\mathbb{R}^n \setminus \{0\}$ of the form
$$\omega_a(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}) = f(a\cdot a)\det(a, \vec{v}_1, \ldots, \vec{v}_{n-1}).$$
   (a) Let $\rho \in SO(n)$ be a rotation of $\mathbb{R}^n$. Show that $\rho^*\omega = \omega$.
   (b) Let $c \in \mathbb{R}_{>0}$ and let $g : \mathbb{R}^n \to \mathbb{R}^n$ be multiplication by the scalar c. Under what condition on f will we have $g^*\omega = \omega$?
   (c) Under what condition on f will we have $d\omega = 0$?

3. This problem begins introducing one of the big tools of differential topology, the de Rham cohomology groups. Let U be an open subset of $\mathbb{R}^n$. Let $\Omega^p(U)$ be the vector space of p-forms on U. Let $Z^p(U)$ be the closed p-forms – those ω with $d\omega = 0$ – and let $B^p(U)$ be the exact p-forms – those ω which are of the form $d\eta$ for some $\eta \in \Omega^{p-1}(U)$.
   (a) Show that $B^p(U) \subseteq Z^p(U)$.

   This allows us to make the definition: $H^p(U)$ is the quotient vector space, $Z^p(U)/B^p(U)$.
   (b) Let $\phi : U \to V$ be a smooth map. Show that ϕ^* takes $B^p(V) \to B^p(U)$ and takes $Z^p(V) \to Z^p(U)$.

   Thus, we can define $\phi^* : H^p(V) \to H^p(U)$.

   Let $\alpha \in H^p(U)$ and $\beta \in H^q(U)$. Lift these forms to $\widetilde{\alpha} \in Z^p(U)$ and $\widetilde{\beta} \in Z^p(U)$.
   (c) Show that $\widetilde{\alpha} \wedge \widetilde{\beta} \in Z^{p+q}(U)$.
   (d) Show that the image of $\widetilde{\alpha} \wedge \widetilde{\beta}$ in $H^p(U)$ does not depend on the choice of lifts of α and β.

   This allows us to define $\wedge : H^p(U) \times H^q(U) \to H^{p+q}(U)$.

4. Let S^1 be the unit circle $\{(x,y) : x^2+y^2 = 1\}$. Let $p : \mathbb{R} \to S^1$ be the map $p(\theta) = (\cos\theta, \sin\theta)$. Let $\phi : S^1 \to S^1$ be a smooth map. In the first parts of this problem, we will discuss the challenge of finding $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ such that $p \circ \tilde{\phi} = \phi \circ p$.

   Choose θ_0 such that $p(\theta_0) = \phi(p(0))$. Define
$$\tilde{\phi}(\theta) = \int_{[0,\theta]} p^*\phi^*(-y dx + x dy) + \theta_0.$$
   In other words, $p^*\phi^*(-y dx + x dy)$ is a 1-form on $\mathbb{R}$, and we integrate it from 0 to θ.
   (a) Show that $p(\tilde{\phi}(\theta)) = \phi(p(\theta))$. (Hint: Differentiate both sides.)
   (b) Show that there is an integer w (the winding number) such that $\tilde{\phi}(\theta+2k\pi) = \tilde{\phi}(\theta)+2kw\pi$.

   Now, let $\phi_0 : S^1 \to S^1$ and $\phi_1 : S^1 \to S^1$ be two maps with the same winding number k. Let $\tilde{\phi}_0$ and $\tilde{\phi}_1$ be the maps constructed above. For $s \in [0,1]$ and $\theta \in \mathbb{R}$, put $\tilde{\phi}(\theta, s) = (1-s)\tilde{\phi}_0(\theta) + s\tilde{\phi}_1(\theta)$.
   (c) Show that there is a map $\phi : S^1 \times [0,1] \to S^1$ such that $\phi(p(\theta), s) = p(\tilde{\phi}(\theta, s))$.
   (d) Show that this map ϕ restricts to ϕ_0 on $S^1 \times \{0\}$ and to ϕ_1 on $S^1 \times \{1\}$.