Problem Set 6 – due Friday March 9

See the course website for policy on collaboration.

1. (a) Show that
$$\left\{ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} : \theta \in \mathbb{R} \right\}$$
is a maximal torus in $SO(3)$. (See the IBL worksheets for definitions and the maximal torus criterion.)

   (b) Describe a maximal torus in $SO(n)$, and prove your answer is correct. Hint: You'll want to consider the cases of n even and n odd separately.

   (c) Let $U(n) \subset \mathrm{GL}_n\mathbb{C}$ be the unitary group, consisting of those matrices with $XX^\dagger = \mathrm{Id}$. Let D be the subgroup of matrices of the form $\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$. Show that D is a maximal torus of $U(n)$.

2. Let X be a topological space. Recall that X is called Hausdorff if, for x and y any two distinct points of X, there are open sets $U \ni x$ and $V \ni y$ with $U \cap V = \emptyset$.

   (a) Let $\Delta \subset X \times X$ be the diagonal $\{(x,x) : x \in X\}$. Show that X is Hausdorff if and only if Δ is closed in the product topology on X.

   (b) Let $U_1, U_2, \dots, U_n$ be an open cover of X. Show that X is Hausdorff if and only if, for all i and j, the set $\{(x,x) : x \in U_i \cap U_j\}$, is a closed subset of $U_i \times U_j$ (again, in the product topology).

3. Let f be a smooth function $\mathbb{R}^n \to \mathbb{R}$. Show that there exist smooth functions $g_1, g_2, \dots, g_n : \mathbb{R}^n \to \mathbb{R}$ such that $f(x_1, \dots, x_n) = f(0) + \sum_i x_i g_i(x_1, \dots, x_n)$. There are several proofs, here is a hint to one of them: $f(x_1, \dots, x_n) - f(0, \dots, 0) = \int_{t=0}^1 \frac{d}{dt} f(tx_1, \dots, tx_n) dt$.

4. Let $\sim$ be the equivalence relation on $\mathbb{R}^n \setminus \{0\}$ where $\vec{v}$ and $\vec{w}$ are equivalent if they are proportional. (In other words, if $\vec{v} = c\vec{w}$ for some nonzero scalar c.) We define $\mathbb{RP}^{n-1}$ to be $(\mathbb{R}^n \setminus \{0\})/\sim$ with the quotient topology. In other words, a subset of $\mathbb{RP}^{n-1}$ is open if and only if its preimage in $\mathbb{R}^n \setminus \{0\}$ is open. In this problem, we will show $\mathbb{RP}^{n-1}$ is a manifold.

   We write $x_1, x_2, \dots, x_n$ for the coordinates on $\mathbb{R}^n$. Let $\tilde{U}_i$ be the open set $\{x_i \neq 0\}$ in $\mathbb{R}^n \setminus \{0\}$ and let U_i be $\tilde{U}_i/\sim$.

   (a) Show that the U_i are an open cover of $\mathbb{RP}^{n-1}$. (This is meant to be easy.)

   (b) Show that U_i is homeomorphic to $\mathbb{R}^{n-1}$. In other words, give inverse continuous maps $U_i \to \mathbb{R}^{n-1}$ and $\mathbb{R}^{n-1} \to U_i$.

   You have now shown that $\mathbb{RP}^{n-1}$ is covered by open set homeomorphic to $\mathbb{R}^{n-1}$. To finish off the problem:

   (c) Show that $\mathbb{RP}^{n-1}$ is Hausdorff.