PROBLEM SET 6 – DUE FRIDAY MARCH 16

See the course website for policy on collaboration.

1. Let X and Y be smooth manifolds of dimension n and let $\phi : X \to Y$ be a smooth map.
   (a) Let $x \in X$. Suppose that the linear map $(D\phi)_x : T_x X \to T_{\phi(x)} Y$ is an isomorphism. Show that there are open neighborhoods U of x and V of $\phi(x)$ such that ϕ maps $U \to V$ bijectively and there is a smooth inverse $V \to U$. (Yes, this is the inverse function theorem. The point is to write down the details of doing this with patches, on manifolds.)
   (b) Suppose that ϕ is a bijection and that, for all $x \in X$, the linear map $D\phi_x : T_x X \to T_{\phi(x)} X$ is an isomorphism. Show that $\phi^{-1} : Y \to X$ is smooth.
2. Let X and Y be smooth manifolds, with smooth atlases $\{(f_i, P_i, U_i)\}$ and $\{(g_j, Q_j, V_j)\}$.
   (a) Define a smooth atlas on $X \times Y$, and prove that the transition maps in your atlas are smooth.
   (b) For $x \in X$ and $y \in Y$, give a natural isomorphism between the vector spaces $T_{(x,y)}(X \times Y)$ and $T_x X \oplus T_y Y$.
3. In this problem, you may use the previous problem even if you haven't done it. Let G be a smooth manifold and let $\mu : G \times G \to G$ be a smooth map making G into a group with identity element e. Put $\mathfrak{g} = T_e G$.

   So we have a map $D\mu_e : T_{(e,e)}(G \times G) \to \mathfrak{g}$ and, by the previous problem, we have a natural isomorphism $\mathfrak{g} \oplus \mathfrak{g} \to T_{(e,e)}(G \times G)$. Composing these gives a linear map $\mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ which, by slight abuse of notation, we will also denote $D\mu_e$.
   (a) Show that $D\mu_e(X, Y) = X + Y$.

   For $g \in G$, let $L_g : G \to G$ be the map $L_g(h) = gh$ and let $R_g : G \to G$ be the map $R_g(h) = hg$.
   (b) Show that $DL_g : \mathfrak{g} \to T_g G$ and $DR_g : \mathfrak{g} \to T_g G$ are isomorphisms.
   (c) When g is GL_n (so $\mathfrak{g}$ is the vector space of $n \times n$ matrices), give simpler formulas for $DL_g X$ and $DR_g X$.
   (d) For g and $h \in G$, and $X \in T_g G$, $Y \in T_h Y$, show that

$$D\mu_{(g,h)}(X, Y) = DL_g(DR_h(DL_g^{-1} X + DR_h^{-1} Y)).$$

4. This question demonstrates why the next question is phrased the way it is.
   (a) Give an example of a manifold X, and open set $V \subset X$ and a smooth function $\phi : V \to \mathbb{R}$ such that there is **no** smooth function $\widehat{\phi} : X \to \mathbb{R}$ such that $\widehat{\phi}(y) = \phi(y)$ for $y \in V$.
   (b) Define a smooth premanifold to be like a smooth manifold, but without the Hausdorff condition. Give an example of a premanifold X, a point $x \in X$ with open neighborhood V, and a smooth function $\phi : V \to \mathbb{R}$, such that there does **not** exist any pair $(\widehat{\phi}, W)$, where $\widehat{\phi} : X \to \mathbb{R}$ is a smooth function, W is an open neighborhood of x, and $\widehat{\phi}(y) = \phi(y)$ for $y \in W$. (Hint: Take X to be the doubled line as a topological space, but use an unusual map $\mathbb{R} \to X$ as a chart around one of the two zeroes.)

One more problem on back.

5. The point of this question is to prove the following lemma: Let X be a smooth manifold; remember that this includes being Hausdorff. Let $V \subset X$ be open, let $\phi : V \to \mathbb{R}$ be smooth and let $x \in V$. Then there exists a smooth function $\widehat{\phi} : X \to \mathbb{R}$, and an open neighborhood $W \ni x$, such that $\widehat{\phi}(y) = \phi(y)$ for $y \in W$.

   Choose a patch $f : P \to U$ with $f(p) = x$ for some $p \in P$. Choose $0 < r_1 < r_2 < r_3$ such that $\overline{B}_p(r_3) \subset P$ and $f(\overline{B}_p(r_3)) \subset V$. (Here $B_p(r)$ is the open ball of radius r around x, and $\overline{B}_p(r)$ is the closed ball.)

   Let $h : \overline{B}_p(r_3) \to \mathbb{R}$ be a smooth function with $h(x) = 1$ for $x \in \overline{B}_p(r_1)$ and $h(x) = 0$ for $x \in \overline{B}_p(r_3) \setminus B_p(r_2)$. Put

$$\widehat{\phi}(y) = \begin{cases} h(f^{-1}(y))\,\phi(y) & y \in f(B_p(r_3)) \\ 0 & y \notin f(\overline{B}_p(r_2)) \end{cases}.$$

   Note that if $y \in f(B_p(r_3)) \setminus f(\overline{B}_p(r_2))$, then $h(f^{-1}(y)) = 0$, so this makes sense.

   (a) Check that $\widehat{\phi}$ is smooth. (Hint: You will need the Hausdorff hypothesis!)

   (b) Put $W = f(B_p(r_1))$. Check that W is open in X, and $\widehat{\phi}(y) = \phi(y)$ for $y \in W$.