Problem Set 6 – due Friday March 23

See the course website for policy on collaboration.

1. In this problem, we show that "connected" and "path connected" are the same for manifolds.
   (a) For X a topological space, and x_0 and $x_1 \in X$, we define $x_0 \sim x_1$ if there is a continuous map $\phi : [0,1] \to X$ with $\phi(0) = x_0$ and $\phi(1) = x_1$. Show that $\sim$ is an equivalence relation.

   The equivalence classes of $\sim$ are called "path connected components" of X.
   Now, let X be a topological manifold.
   (b) Verify that the path connected components of X are open in X.
   (c) Verify that the path connected components of X are closed in X.

2. In this problem, we will explain why we don't need to include smoothness of the inverse in the axioms of a Lie group. You may wish to look at Problem 3 on the previous problem set. Let G be a smooth manifold and let $\mu : G \times G \to G$ be a smooth map making G into a group. Define $\tau : G \times G \to G \times G$ by
$$\tau(g,h) = (g, gh).$$
   (a) Show that τ is bijective and the linear map $D\tau_{(g,h)}$ is an isomorphism for all g and h.
   (b) Conclude that τ^{-1} is a smooth map; deduce that the map $g \mapsto g^{-1}$ from $G \to G$ is smooth.

3. In our definition of a manifold, we started with a topological space X and a bunch of patches $f : P \to U \subset X$. Sometimes, we are given X only as a set and want to use the patches to define the topology. This problem spells out the details. To make it shorter, we do the topological manifold version; the smooth version is analogous.

   Let X be a set, and let U_i be subsets of X with $X = \bigcup_i U_i$. Let P_i be subsets of $\mathbb{R}^d$ and let $f_i : P_i \to U_i$ be a bijection.

   Suppose that, for all i and j, the set $f_i^{-1}(U_i \cap U_j)$ is open in P_i, and the map $f_j^{-1} \circ f_i : f_i^{-1}(U_i \cap U_j) \longrightarrow f_j^{-1}(U_i \cap U_j)$ is a homeomorphism. Define $V \subset X$ to be open if and only if, for all V, the set $f_i^{-1}(V)$ is open in P_i.
   (a) Show that this is a topology on X.
   (b) Show that the map $f_i : P_i \to U_i$ is a homeomorphism, in the subspace topology on X.

4. We finally get around to checking a claim from class a few weeks ago: Let V be a finite dimensional real vector space. Let $K \subset V$ be a subgroup (under addition) which is discrete (meaning that, for any $g \in K$, there is an open set $U \subset V$ with $U \cap K = \{g\}$). We show that $K = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k$ for some linearly independent set $\{v_1, \ldots, v_k\}$.

   Our proof is by induction on $\dim V$, and we note that the theorem is obvious when $K = \{0\}$. So we assume $\dim V > 0$ and $K \neq \{0\}$ from now on.

   Choose a positive definite dot product at V, and let v_k minimize $v \cdot v$ over all nonzero $v \in K$. Put $V' = V/\mathbb{R}v_k$ and $K' = K/\mathbb{Z}v_k$.
   (a) Show that K' injects into V'.
   (b) Show that K' is a discrete subgroup of V'. (This is meant to be a bit challenging.)
   By induction, $K' = \mathbb{Z}v'_1 \oplus \cdots \oplus \mathbb{Z}v'_{k-1}$ for some linearly independent $v'_1, \ldots, v'_{k-1}$ in V'.
   (c) Explain how to finish the proof and show $K = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k$ for some linearly independent set $\{v_1, \ldots, v_k\}$.

One more challenging problem on back!

5. Let H be the Lie group of matrices of the form $\left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$. Let Z be the subgroup $\left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : z \in \mathbb{Z} \right\}$. In this problem, we will prove that H/Z is not a subgroup of any $\mathrm{GL}_N(\mathbb{R})$.

   (a) Check that Z is a normal subgroup of H, so H/Z is a group.

   Let $\bar{\rho} : H/Z \to \mathrm{GL}_N(\mathbb{R})$ be a map which is both smooth and a group homomorphism. Our goal is to show that $\bar{\rho}$ is not injective. Let ρ be the composite $H \to H/Z \to \mathrm{GL}_n(\mathbb{R})$. Let $\mathfrak{h}$ be the Lie algebra of H, with basis $e = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$. So $D\rho_{\mathrm{Id}}$ maps $\mathfrak{h} \to \mathfrak{g}$; abbreviate $D\rho_{\mathrm{Id}} = \sigma$.
   Recall from the November 17 IBL that $\rho(e^X) = e^{\sigma(X)}$ and $[\sigma(X), \sigma(Y)] = \sigma([X, Y])$.

   (b) Show that $\sigma(g)$ is diagonalizable with eigenvalues of the form $2\pi i k$.

   Let m_k be the multiplicity of $2\pi i k$ as an eigenvalue of σg. We change bases so that $\sigma(g)$ is block diagonal, with blocks $(2\pi i k)\mathrm{Id}_{m_k}$.

   (c) Show that $\sigma(e)$ and $\sigma(f)$ are block diagonal, with blocks of size $m_k \times m_k$.

   We'll call these blocks E_k and F_k.

   (d) Show that $[E_k, F_k] = (2\pi i k)\mathrm{Id}_{m_k}$.

   (e) Show that $m_k = 0$ for $k \neq 0$. (Hint: This uses the previous part, and a clever little trick.)

   (f) Deduce, finally, that $\bar{\rho}$ is not injective.