Problem Set 9 – due Friday March 30

See the course website for policy on collaboration.

1. In class, we computed that S^2 could be covered by two patches U_1 and U_2 parametrized by $P_1 = P_2 = \mathbb{R}^2$. Writing (u_1, v_1) for coordinates on P_1, (u_2, v_2) for coordinates on P_2 and (x, y, z) for coordinates on S^2, the parametrizations are

$$\begin{aligned} f_1(u_1, v_1) &= \left(\tfrac{2u_1}{1+u_1^2+v_1^2}, \tfrac{2v_1}{1+u_1^2+v_1^2}, \tfrac{1-u_1^2-v_1^2}{1+u_1^2+v_1^2} \right) \\ f_2(u_2, v_2) &= \left(\tfrac{2u_2}{1+u_2^2+v_2^2}, \tfrac{2v_2}{1+u_2^2+v_2^2}, \tfrac{-1+u_2^2+v_2^2}{1+u_2^2+v_2^2} \right). \end{aligned}$$

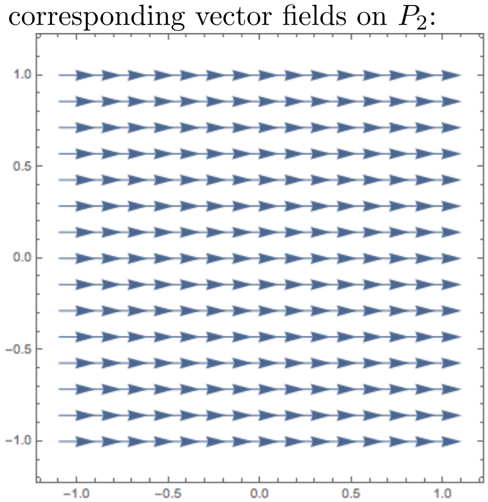
with transition function

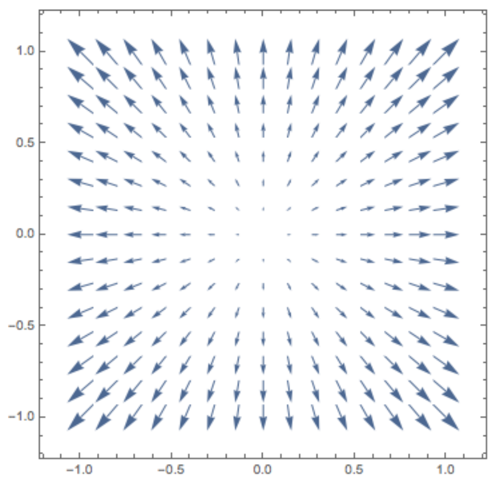
$$(f_2^{-1} \circ f_1)(u_1, v_1) = \left(\frac{u_1}{u_1^2 + v_1^2}, \frac{v_1}{u_1^2 + v_1^2} \right).$$

(a) In patch P_1, at the point (u_1, v_1), let $\vec{x}$ be the vector $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Find the corresponding tangent vector to S^2 at $f_1(u_1, v_1)$, and then the corresponding vector in P_2.

(b) Repeat the previous computation for the vector which is $\left[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right]$ at the point (u_1, v_1) of P_1.

(c) The figures below sketch the vector fields on P_1 in the previous two parts. Sketch the corresponding vector fields on P_2:





(d) On problem set 3, you computed that $\omega = x dy \wedge dz - y dx \wedge dz + z dx \wedge dy$ is a rotationally invariant 2-form on S^2. Compute $f_1^* \omega$.

2. In general, second derivatives of functions on manifolds aren't easy to talk about. But they are easy at critical points! This question explains.

Let X be a smooth d-fold and let $g : X \to \mathbb{R}$ be a smooth function. Let $x \in X$ and suppose that $(Dg)_x = 0$.

(a) Let $\gamma(t)$ be a smooth curve in X (in other words, smooth map $(-\epsilon, \epsilon) \to X$ with $\gamma(0) = x$. Show that $\lim_{t \to 0} \Big(g(\gamma(t)) - g(x) \Big) / t^2$ exists.

(b) Show that the limit in the previous problem only depends on the vector $\gamma'(0)$ in $T_x X$.

(c) Show that $\gamma'(0) \mapsto \lim_{t \to 0} \Big(g(\gamma(t)) - g(x) \Big) / t^2$ is a quadratic form, in the sense of Problem 4 on Math 395 Problem Set 4.

This quadratic form is called the Hessian, and serves as a notion of second derivative which you can define at critical points without any coordinates.

3. The Grassmannian, $G(d,n)$, is the set of d-dimensional vector subspaces of $\mathbb{R}^n$. In this problem, we will put the structure of a smooth manifold on $G(d,n)$.

   Let X and Y be subspaces of $\mathbb{R}^n$ of dimensions d and $n-d$ with $\mathbb{R}^n = X \oplus Y$ and let $\phi : X \to Y$ be a linear map. Define $\Gamma(\phi)$ (the graph of ϕ) to be $\{x + \phi(x) : x \in X\} \subset \mathbb{R}^n$. Define $U(X,Y) \subset G(d,n)$ to be the set of d-planes of the form $\Gamma(\phi)$ for some $\phi : X \to Y$.

   (a) Explain a bijection $f_{X,Y} : \mathbb{R}^{d(n-d)} \to U(X,Y)$. You'll be using this bijection in the rest of the problem, so choose a reasonable one!

   (b) Show that, for any two splittings (X_1,Y_1) and (X_2,Y_2), the set $f^{-1}_{(X_1,Y_1)}\Big(U(X_1,Y_1) \cap U(X_2,Y_2)\Big)$ is open in $\mathbb{R}^{d(n-d)}$.

   (c) Show that the map $f^{-1}_{X_2,Y_2} \circ f_{X_1,Y_1}$ from $f^{-1}_{(X_2,Y_2)}\Big(U(X_1,Y_1) \cap U(X_2,Y_2)\Big)$ to $f^{-1}_{(X_2,Y_2)}\Big(U(X_1,Y_1) \cap U(X_2,Y_2)\Big)$ is smooth.

   We can now use the open cover $U(X,Y)$ and the bijections $f_{X,Y}$ to put a topology on $G(d,n)$.

   (d) Show that $G(d,n)$ is Hausdorff in this topology.

   (e) Show that $G(d,n)$ is compact. (Hint: Every d-plane has an orthonormal basis.)