EVERY SUBSPACE HAS A BASIS

Let V be a subspace of $\mathbb{R}^n$. We never actually proved that V has a basis; we only showed that, if V is an image or a kernel of a linear map, then it has one. For example, if we have linear maps $A : \mathbb{R}^m \to \mathbb{R}^n$ and $B : \mathbb{R}^n \to \mathbb{R}^p$, then $\mathrm{Im}(A) \cap \mathrm{Ker}(B)$ is a subspace, but we didn't prove it has a basis. This note corrects that.

Since V is contained in $\mathbb{R}^n$, there can be no more than n linearly independent vectors in V. Let m be the largest number such that there are m linearly independent vectors in V, and let $\vec{v}_1$, $\vec{v}_2$, …, $\vec{v}_m$ be a list of such vectors.

We claim that $\vec{v}_1$, $\vec{v}_2$, …, $\vec{v}_m$ is a basis for V. By definition, the $\vec{v}_i$ are linearly independent, so we must show that they span V.

Consider any $\vec{w}$ in V. Since there are not $m+1$ linearly independent vectors in V, there must be a linear relation:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m + b\vec{w} = 0.$$

Since the vectors $\vec{v}_i$ are linearly independent, the scalar b is nonzero. So we have

$$\vec{w} = -\frac{c_1}{b}\vec{v}_1 - \frac{c_2}{b}\vec{v}_2 - \cdots - \frac{c_m}{b}\vec{v}_m.$$

This shows that $\vec{w}$ is in the span of the $\vec{v}_i$, as promised.