A QUICK PROOF OF THE CAUCHY-SCHWARTZ INEQUALITY

Let u and v be two vectors in $\mathbb{R}^n$. The Cauchy-Schwartz inequality states that

$$|u \cdot v| \leq |u||v|.$$

Written out in coordinates, this says

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \leq \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \quad (*).$$

This equation makes sure that vectors act the way we geometrically expect. For example, we know that

$$|u+v|^2 = (u+v) \cdot (u+v) = u \cdot u + v \cdot v + 2u \cdot v = |u|^2 + |v|^2 + 2u \cdot v.$$

Using Cauchy-Schwartz, we have

$$|u|^2 + |v|^2 + 2u \cdot v \leq |u|^2 + |v|^2 + 2|u||v| = (|u| + |v|)^2.$$

So the Cauchy-Schwartz inequality tells us that

$$|u+v|^2 \leq (|u| + |v|)^2 \quad \text{or} \quad |u+v| \leq |u| + |v|.$$

In other words, the length of the sum of two vectors is no more then the sum of the lengths of the vectors.

As explained in class, if you believe that vectors in hundreds of dimensions act like the vectors you know and love in $\mathbb{R}^2$, then the Cauchy-Schwartz inequality is a consequence of the law of cosines. Specifically, $u \cdot v = |u||v| \cos\theta$, and $\cos\theta \leq 1$. In case you are nervous about using geometric intuition in hundreds of dimensions, here is a direct proof.

First, note that we have

$$w \cdot w = w_1^2 + w_2^2 + \cdots w_n^2 \geq 0$$

for any w.

We rescale u and v to new vectors which have the same length; namely $|v|u$ and $|u|v$. We take the difference of these two vectors: $|u|v - |v|u$. So this is a vector which vanishes if u is a positive multiple of v. We compute the dot product of this vector with itself:

$$0 \leq (|u|v - |v|u) \cdot (|u|v - |v|u) = |u|^2(v \cdot v) - 2|u||v|(u \cdot v) + |v|^2(u \cdot u) = 2|u|^2|v|^2 - 2|u||v|(u \cdot v).$$

Rearranging

$$\begin{aligned} 2|u||v|(u \cdot v) &\leq 2|u|^2|v|^2 \\ u \cdot v &\leq |u||v| \end{aligned}$$

A similar argument using $(|u|v + |v|u) \cdot (|u|v + |v|u)$ shows that $-u \cdot v \leq |u||v|$ as well. So

$$|u \cdot v| \leq |u||v|$$

as we promised.