INJECTIVE, SURJECTIVE AND INVERTIBLE

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Surjectivity: Maps which hit every value in the target space

Let's start with a puzzle. I have a remote control car, controlled by 3 buttons. When I hold down the red button, it moves in direction $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; when I hold down the green button it moves in direction $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$; when I hold down the blue button, it moves in direction $\begin{pmatrix} -3 \\ -5 \end{pmatrix}$. Can I get anywhere in the plane? For example, can I get to $\begin{pmatrix} 12 \\ 19 \end{pmatrix}$?

Let r be the amount of time I hold down the red button, and write g and b for the green and blue buttons. So we move

$$\begin{pmatrix} 1 & 2 & -5 \\ 2 & 3 & -3 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix}$$

and we want to solve

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & -5 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 12 \\ 19 \end{pmatrix}.$$

We run through the usual row reduction process

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

So there are lots of solutions, the simplest of which is to hold down the red button for 2 seconds and the green for 5.

Notice that, if I wanted to move to $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ instead, I only need to redo the computations on the right hand side of the equations; the left hand sides stay the same.

For *any* $\begin{pmatrix} x \\ y \end{pmatrix}$, we can find[1] (r, g, b) values which will move us in direction $\begin{pmatrix} x \\ y \end{pmatrix}$. There is a term for this:

Vocabulary. *A linear map $A : \mathbb{R}^k \to \mathbb{R}^\ell$ is called* surjective *if, for every v in $\mathbb{R}^\ell$, we can find u in $\mathbb{R}^k$ with $A(u) = v$.*

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[1]From the physical motivation from this problem, in only makes sense to look at solutions where r, g and $b \geq 0$. In fact, such solutions exist in this case. The subject of solving linear equations together with inequalities is studied in Math 561. I'll ignore this issue.

Another word which is sometimes used is *onto*.

So we say that $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & -5 \end{pmatrix}$ is surjective.

Let's say a new car comes on the market. It moves by the matrix $\begin{pmatrix} 1 & 4 & -2 \\ 3 & 12 & -6 \end{pmatrix}$. Can we still go anywhere?

Let's run the row reduction algorithm again. We want to move to position $\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} 1 & 4 & -2 \\ 3 & 12 & -6 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} x \\ y - 3x \end{pmatrix}.$$

So we can only go to $\begin{pmatrix} x \\ y \end{pmatrix}$ if $y - 3x = 0$. In other words, this car can only drive along the line $y = 3x$.

The map $\begin{pmatrix} 1 & 4 & -2 \\ 3 & 12 & -6 \end{pmatrix}$ is *not* surjective.

Let's understand the difference between these two examples:

General Fact. *Let A be a matrix and let A_{red} be the row reduced form of A. If A_{red} has a leading 1 in every row, then A is surjective. If A_{red} has an all zero row, then A is not surjective.*

Remember that, in a row reduced matrix, every row either has a leading 1, or is all zeroes, so one of these two cases occurs.

INJECTIVITY: MAPS THAT DON'T DESTROY INFORMATION

Wanda owns two types of pets: birds and cats. I ask her how many of each she has. She replies:

"My pets have 14 legs, 10 eyes and 5 tails."

Can we figure out how many of each animal there are?

Let b be the number of birds and c the number of cats. So

$$\begin{pmatrix} 2 & 4 \\ 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \\ 5 \end{pmatrix}.$$

We proceed as usual

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

So there are 3 birds and 2 cats.

Yes, Wanda has given us enough clues to recover the data.

On the other hand, suppose Wanda said

"My pets have 5 heads, 10 eyes and 5 tails."

Then we get

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \quad .$$

All we can conclude is that the total number of pets is 5; we can't tell how many are cats and how many are birds. Wanda has wickedly failed to give us enough information!

Vocabulary. *A linear map $A : \mathbb{R}^k \to \mathbb{R}^\ell$ is called* injective *if, for every v in $\mathbb{R}^\ell$, there is at most one u in $\mathbb{R}^k$ with $A(u) = v$.*

In other words, A does preserves enough data to recover u. Another word which is sometimes used is *one to one*.

So $\begin{pmatrix} 2 & 4 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}$ is injective but $\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}$ is not.

We have discussed before how, if there are columns without leading 1's, they give us multiple solutions. If, on the other hand, every column has a leading 1 in it, then there is at most one solution. In our new language:

General Fact. *Let A be a matrix and let A_{red} be the row reduced form of A. If A_{red} has a leading 1 in every column, then A is injective. If A_{red} has a column without a leading 1 in it, then A is not injective.*

INVERTIBLE MAPS

If a map is both injective and surjective, it is called *invertible*. This means, for every v in $\mathbb{R}^\ell$, there is *exactly one* solution to $Au = v$. So we can make a map back in the other direction, taking v to u.

Note that, if A is invertible, then A_{red} has a 1 in every column and in every row. This can only happen if A is a square matrix, so $k = \ell$.

This reverse map is called A^{-1}. We have

$$AA^{-1} = A^{-1}A = \text{Id}_k.$$