The point of this note is to prove the following fact:

Theorem If A is a square matrix, and $AB = \mathrm{Id}$, then $BA = \mathrm{Id}$.

A similar argument, which I'll leave for you to work out, shows that

Theorem If A is a square matrix, and $BA = \mathrm{Id}$, then $AB = \mathrm{Id}$.

So suppose that A is a square matrix and $AB = \mathrm{Id}$.

Claim 1 A is surjective.

Proof Consider any vector $\vec{y}$. I must find a vector $\vec{x}$ such that $A\vec{x} = \vec{y}$. I claim that taking $\vec{x} = B\vec{y}$ does the trick. Let's check:

$$A\vec{x} = A(B\vec{y}) = (AB)\vec{y} = \mathrm{Id}\vec{y} = \vec{y}$$

as promised. $\square$

As discussed in class, the fact that A is surjective means that the row reduction of A has a leading 1 in every row. Since A is square, it has the same number of rows and columns, and therefore has a leading 1 in every column. This, in turn, means that A is injective. So we have established:

Claim 2 A is injective.

For any vector $\vec{v}$, we have

$$A(BA\vec{v}) = (AB)A\vec{v} = \mathrm{Id}A\vec{v} = A\vec{v}.$$

In other words, writing $\vec{w} = BA\vec{v}$, we have $A\vec{w} = A\vec{v}$. But we just checked that A is injective. So this shows us that

$$BA\vec{v} = \vec{v}.$$

We have shown that, for all vectors $\vec{v}$, we have $BA\vec{v} = \vec{v}$. So $BA = \mathrm{Id}$.