Complex inner products

Let V be a vector space over $\mathbb{C}$. Our notions of ***bilinear form*** and ***symmetric bilinear form*** still make sense:

$$B(\vec{v}_1 + \vec{v}_2, \vec{w}) = B(\vec{v}_1, \vec{w}) + B(\vec{v}_2, \vec{w})$$

$$B(\vec{v}, \vec{w}_1 + \vec{w}_2) = B(\vec{v}, \vec{w}_1) + B(\vec{v}, \vec{w}_2)$$

$$B(c\vec{v}, \vec{w}) = B(\vec{v}, c\vec{w}) = cB(\vec{v}, \vec{w}).$$

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However, it doesn't make sense to ask for this form to be positive definite: If $B(\vec{v}, \vec{v}) > 0$ then $B(i\vec{v}, i\vec{v}) = i^2 B(\vec{v}, \vec{v}) = -B(\vec{v}, \vec{v}) < 0$.

If we stick with the naive dot product, it is not true that subspaces over $\mathbb{C}$ have orthonormal bases: Every vector in $\mathbb{C}\left[\begin{smallmatrix}1\\i\end{smallmatrix}\right]$ has length 0.

And it is not true that symmetric matrices are diagonalizable: $\left[\begin{smallmatrix}1 & i\\i & -1\end{smallmatrix}\right]$ has characteristic polynomial x^2, but only has 1-dimensional 0-eigenspace.

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We can fix these problems by introducing ***sesquilinear forms***.

Vocabularly interlude: "Sesqui-" is a prefix meaning "one and a half", in the same way that "bi-" means "two".

These forms are linear in their second entry and "halfway linear" in their first entry.

Sesquilinear forms

Let V be a complex vector space. A ***sesquilinear form*** takes an input two vectors from V and gives as output a complex number such that:

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$$B(\vec{v}, c\vec{w}) = cB(\vec{v}, \vec{w}) \qquad B(c\vec{v}, \vec{w}) = \overline{c}B(\vec{v}, \vec{w}).$$

Here $\overline{c}$ is the ***complex conjugate***: $\overline{c_1 + c_2 i} = c_1 - c_2 i$.

Notice that we now have

$$B(i\vec{v}, i\vec{v}) = i\bar{i}B(\vec{v}, \vec{v}) = i(-i)B(\vec{v}, \vec{v}) = B(\vec{v}, \vec{v}).$$

What does this look like in matrices? Given an $m \times n$ complex matrix A, we define $A^\dagger$ to be the $n \times m$ matrix $\overline{A}^T$ – we get the transpose, and we take the complex conjugate of each entry.

A sequilinear form on $\mathbb{C}^n$ is given by an $n \times n$ complex matrix Q, with $B(\vec{x}, \vec{y}) = \vec{x}^\dagger Q \vec{y}$.

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Note: Your book uses the opposite convention about which variable is linear, writing $B(\vec{v}, c\vec{w}) = \overline{c} B(\vec{v}, \vec{w})$ and $B(c\vec{v}, \vec{w}) = cB(\vec{v}, \vec{w})$. As a result, they are constantly writing $(\vec{x}|\vec{y}) = \vec{y}^\dagger Q \vec{x}$. Both are roughly equally common in the literature; taking the first variable linear (like your book) is more common in pure math; taking the second variable linear is more common in applied math, physics and engineering.

Also, your book uses $*$ instead of $\dagger$. But your book also uses $*$ for the dual space!

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A sesquilinear form is ***Hermitian*** if

$$B(\vec{v}, \vec{w}) = \overline{B(\vec{w}, \vec{v})}.$$

Note that, for a Hermitian linear form, we have $B(\vec{v}, \vec{v}) = \overline{B(\vec{v}, \vec{v})}$, so $B(\vec{v}, \vec{v})$ is always real.

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A Hermitian bilinear form is ***positive definite*** if, for all $\vec{v} \neq 0$, we have $B(\vec{v}, \vec{v}) > 0$.

As matrices, Hermitian means that $Q = Q^\dagger$. For example, here is a Hermitian matrix:

$$\begin{bmatrix} 3 & 5+3i & -2+3i \\ 5-3i & 7 & -4i \\ -2-3i & 4i & 11 \end{bmatrix}$$

The most standard positive definite Hermitian form on $\mathbb{C}^n$ is simply:

$$\langle (p_1, p_2, \ldots, p_n),\ (q_1, q_2, \ldots, q_n) \rangle = \overline{p_1} q_1 + \overline{p_2} q_2 + \cdots + \overline{p_n} q_n.$$

In other words,

$$\langle (x_1 + y_1 i, \ldots, x_n + y_n i),\ (u_1 + v_1 i, \ldots, u_n + v_n i) \rangle = \\ (x_1 - y_1 i)(u_1 + v_1 i) + \cdots + (x_n - y_n i)(u_n + v_n i).$$

In particular,

$$\langle (x_1 + y_1 i, \ldots, x_n + y_n i),\ (x_1 + y_1 i, \ldots, x_n + y_n i) \rangle = \\ (x_1 - y_1 i)(x_1 + y_1 i) + \cdots + (x_n - y_n i)(x_n + y_n i) = \\ x_1^2 + y_1^2 + x_2^2 + y_2^2 + \cdots + x_n^2 + y_n^2 \geq 0.$$

Let V be a complex vector space with a positive definite Hermitian form:

Given an orthogonal basis $\vec{u}_1$, $\vec{u}_2$, ..., $\vec{u}_k$ for a subspace L of V, orthogonal projection onto L is given by

$$p_L(\vec{v}) = \sum \frac{B(\vec{u}_i, \vec{v})}{B(\vec{u}_i, \vec{u}_i)} \vec{u}_i$$

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The Gram-Schmidt algorithm works exactly as before. So every finite dimensional vector space with a positive definite Hermitian form has an orthonormal basis.