Let V be a complex vector vector space equipped with a positive definite Hermitian form $B(\ , \ )$. We check some of the properties of $B(\ , \ )$ which were asserted without proof on Tuesday:

Problem 1. Let $\vec{v} \in V$. You might wonder whether we should define $\vec{v}^{\perp}$ to be $\{\vec{w} : B(\vec{v}, \vec{w}) = 0\}$ or $\{\vec{w} : B(\vec{w}, \vec{v}) = 0\}$. Show that it doesn't matter, because these are the same subspace.

Problem 2. Let W be a subspace of V. Show that $W \cap W^{\perp} = \{\vec{0}\}$.

Recall that $A : V \to V$ is called:

- ***Hermitian*** if $A = A^{\dagger}$
- ***unitary*** if $A^{-1} = A^{\dagger}$.
- ***normal*** if $AA^{\dagger} = A^{\dagger}A$.

Problem 3. Let V be $\mathbb{C}^2$ with the standard Hermitian inner product. Let $A = \left[\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right]$.

(1) Show that A is normal.
(2) Compute an orthonormal eigenbasis for A.
(3) Is A Hermitian? Is it unitary?

Problem 4. Let V be a vector space with a positive definite Hermitian form. Let $T : V \to V$ be a linear operator.

(1) If T is normal, show that there are Hermitian operators X and Y with $T = X + iY$ and $XY = YX$.
(2) Conversely, show that, if X and Y are Hermitian operators X and Y with $T = X + iY$ and $XY = YX$ then T is self-adjoint.

We now come back to some old problems from homework:

Problem 5. On Problem Set 10, Problem 3, we considered the vector space of functions $[-\pi, \pi] \to \mathbb{R}$ with the inner product $\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$. Now that we know about Hermitian forms, consider instead complex valued functions $[-\pi, \pi] \to \mathbb{C}$ with the inner product $\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} \overline{f(x)}g(x)dx$.

(1) Recall that $e^{i\theta} = \cos(\theta) + i \sin(\theta)$. Show that the list of functions $\frac{1}{\sqrt{2\pi}} e^{in\theta}$, for $n \in \mathbb{Z}$, is orthonormal.
(2) Convert the change of basis matrix between this orthonormal basis, and the orthonormal basis $\frac{1}{\sqrt{2\pi}}$, $\frac{1}{\sqrt{\pi}} \cos(n\theta)$, $\frac{1}{\sqrt{\pi}} \sin(n\theta)$ on the Problem Set.

Problem 6. On Problem Set 8, Problem 3, we considered a differential equation for a collection of masses connected by springs. For simplicity, we take all the masses to be equal to m. Let k_{ij} be the spring constant on the spring joining mass i to mass j. Then the equations of motion are

$$m \frac{d^2 x_i(t)}{(dt)^2} = \sum_j k_{ij}(x_j(t) - x_i(t)).$$

(1) Rewrite this equation in the form $m \frac{d^2}{(dt)^2} \vec{x}(t) = -K\vec{x}(t)$ for a matrix K. As a starting example, you might want to do the case where there is a spring from mass 1 to mass 2 of strength k_{12} and a spring from mass 2 to mass 3 of strength k_{23}.
(2) Show that the eigenvalues of K are nonnegative real integers.
(3) Show how to convert eigenvalues of K into solutions of the form $\vec{x}(t) = \vec{a} \cos(\omega t)$ for some vector $\vec{a}$.