Dual spaces

Eigenalues, eigenvectors,
primary decomposition theorem
…

Sesquilinear forms, Hermitian
matrices, normal …

V a vector space over a field F

The dual space, V*, is the set of F-linear maps V —> F.

Given v* in V* and v in V, I can evaluate v* on v to get a scalar v*(v).

If V has a basis e1, e2, e3, …

then, for each i, there is a linear map ei* : V* —> F where

ei*(v) is the coefficient of ei when we write v in the basis e1, e2, ..

Notice that changing the basis e1, e2, e3, … changes the function ei*. Even if we keep ei the same, ei* changes when we change the other basis elements.

If V is finite dimensional, then the ei* are a basis for V*.
In particular, if dim V is finite, then dim V = dim V*.

If A : V —> W is a linear map, then we get a linear map A*: W* —> V* where

(A*(w*))(v) = w*(A(v)).

We have (AB)* = B* A*.

If e1, e2, … em, is a basis for V and f1, f2, …, fn is a basis for W, and A is the matrix of the transformation in this basis, then the matrix for A* in the basis e1*, e2*, … em* and f1*, f2*, …, fn* is A^T.

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Then

$$e_1^* \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = -2$$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = -2e_1 - e_2 + 4e_3$$

For finite dimensional vector spaces,

A* is surjective <—> A is injective
A is surjective <—> A* is injective

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$e_1^* \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x$$

If we want to write v in the basis e1, e2, e3, we need to solve the linear equations

c1 e1 + c2 e2 + c3 e3 = v.

Once we solve them, we'll have ei*(v) = ci.

Let V be the vector space of polynomials of degree <= d.

V has a basis x^0, x^1, …, x^d. Let e0*, e1*, …, ed* be the basis dual to this.

Choose d+1 points p0, p1, .., p_{d+1} and let f_j be evaluation at p_j. Find a formula for e_i* in terms of f_j.

$$\begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{v}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix}^{-1} \vec{v}$$

The ei* are the rows of the matrix [e1 e2 e3]^{-1}.

Dual spaces and inner products:

A bilinear form is a map

$B : V \times V \longrightarrow F$

which is linear in each input.

$B(u1+u2, v) = B(u1, v) + B(u2, v)$
$B(u, v1+v2) = B(u, v1) + B(u, v2)$
$B(cu, v) = B(u, cv) = B(u,v)$

If V is a vector space with an inner product, then that inner product gives an isomorphism between V and V*. In that setting, we often don't distinguish between V and V*.

So, given $A: V \longrightarrow W$,
we'll talk about A^* being the map $W \longrightarrow V$ defined by
$\langle Av, w\rangle = \langle v, A^* w\rangle$.

Again, in orthonormal bases, this is given by the transpose matrix.

This is the same as a linear map from $V \longrightarrow V^*$.

Given B(,) , a bilinear form, and given a vector v in V,

B(v,w) is a linear function of w.

So B(v, ___) is an element of V*.

In reverse, if I have a map

$f : V \longrightarrow V^*$, then

$B(u,v) = f(u)(v)$ is a bilinear form.

A symmetric bilinear form corresponds to a linear map $V \longrightarrow V^*$ which obeys

$(f(u))(v) = (f(v))(u)$.

In matrices, this means a symmetric matrix.

Positive definite means symmetric and $f(v)(v) > 0$ for v != 0.

If we use an inner product to identify V with V*, then a basis of V is orthonormal iff ei and ei* are identified with each other.

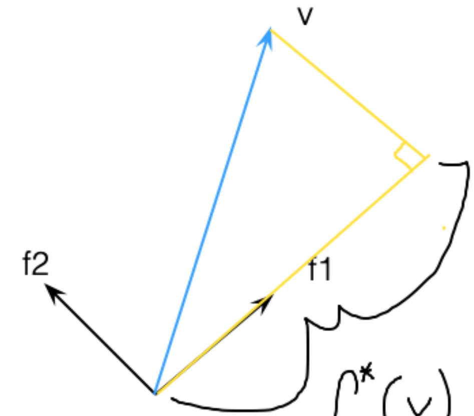
If V is finite dimensional, then dim V* = dim V, so V and V* are isomorphic. An inner product gives a particular isomorphism $f: V \longrightarrow V^*$.

$V = \mathbb{R}^2$

standard inner product.
I am using this inner product to identify V and V*.

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $e_1^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$f_1 = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$ $f_2 = \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}$ $f_1^* = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$ $f_2 = \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}$



$f_1^*(v) = \langle f_1, v \rangle$

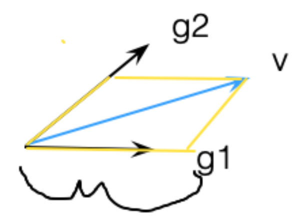
Because f1 and f2 are orthonormal, we have f1* = f1 and f2* = f2. In other words, we can compute the f1 component of v by computing <f1, v>.

$g_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $g_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$g_1^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $g_2^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = (x-y)g_1 + y g_2$

$g_1^* \begin{bmatrix} x \\ y \end{bmatrix} = (x-y) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$



$g_1^*(v) \neq \langle g_1, v \rangle$

Since this basis was not orthonormal, g1* != g1.

A : V —> V and if A(v) = cv for some vector v and scalar c, then v is called an eigenvector and c is called the eigenvalue.

We say that c is an eigenvalue of A if there is a nonzero eigenvector v with eigenvalue c.

We compute the eigenvalues as the roots of the char. poly.

For each root c of the char. poly, we compute the c-eigenspace as as Ker(A - c Id).

An eigenbasis is a basis of eigenvectors.

If A has an eigenbasis, then we say that A is diagonalizable.

When we write A in that basis, we get a diagonal matrix.

If S is the change of basis matrix, we have

A = S D S^{-1}.

Eigenvectors for distinct eigenvalues are always linearly independent. So, if we have n distinct eigenvalues (in other words, if the char. poly. factors into n distinct linear factors) then we have an eigenbases.

Also, real symmetric matrices always have orthonormal eigenbases. As do complex Hermitian matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is diagonal but doesn't have full rank.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is diagonal but doesn't have distinct eigenvalues.

$$A = \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}}_{S} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}^{-1}$$

eigenvector

eigenvalue

Going from factorizations of char poly to matrix forms.

First of all, if char poly factors into linears $(x-c1)(x-c2) \dots (x-cn)$

with the ci distinct, then A is diagonalizable, can choose basis where

$$A = \begin{bmatrix} \boxed{c_1} & & \\ & \ddots & \\ & & \boxed{c_n} \end{bmatrix}$$

If $f(A) = 0$ for any polynomial $f(x)$, and $f(x)$ factors as $g1(x)\, g2(x) \dots gr(x)$ with $GCD(gi, gj) = 1$, then we get a block decomposition of A.

Then we can chooose a basis where

$$A = \begin{bmatrix} \boxed{B_1} & & & \\ & \boxed{B_2} & & \\ & & \ddots & \\ & & & \boxed{B_r} \end{bmatrix}$$

$g_i(B_i) = 0$.

Compute bases for $Ker(g_i(A))$, and put them together to make this basis.

In particular, if char poly factors as g1 g2 .. gr with $GCD(gi, gj) = 1$ then we can find this block form and gi will be the char poly of Bi.

Problem Set 10, Problem 1:

If char poly factors into linears $(x-c1)^{n1} (x-c2)^{n2} \dots (x-cr)^{nr}$ then we can take a basis where B_i is upper triangular with diagnaonl entries ci.

Also on Problem Set 10, Problem 2: If g_i is irreducible, and has degree = size of block B_i, then we can take B_i to look like:

$$\begin{bmatrix} 0 & & 0 & * \\ 1 & 0 & & * \\ & 1 & \ddots & \vdots \\ 0 & & 0 & * \\ & & 1 & * \end{bmatrix}$$

Sesquilinear forms:

V a complex vector space

B : V x V —> C is sesquilinear if

B(u1+u2, v) = B(u1, v) + B(u2, v)
B(u, v1+v2) = B(u, v1) + B(u, v2)
B(u,cv) = c B(u,v)
B(cu, v) = \bar{c} B(u,v)

complex conjugate of c.

A sesquilinear form is Hermitian (analogue of symetric) if

$$B(u,v) = \overline{B(v,u)}$$

$$\left\langle \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right\rangle =$$

$$\sum \overline{u_i} v_i .$$

For a Hermitian form, B(v,v) is always real. We call B(,) positive if B is Hermitian and B(v,v) > 0 for all v != 0.

In a complex vector space with a positive definite Hermitian form, every subspace has an orthonormal basis. You can compute it with the Gram-Schmidt algorithm, just remembering to put complex conjugates in.

If e1, e2, …, ek are orthogonal, the formula for orthogonal projection onto Span(e1, e2, …, ek) is still

$$p(v) = \sum \frac{B(\underline{e_i, v})}{B(e_i, e_i)} e_i .$$

If V and W are both complex vector spaces with postive Hermitian forms, and A : V —> W is a map, then we get a map A^{dagger} : W —> V

$$\underbrace{\langle v, A^{\dagger}(w) \rangle}_{\text{form on w}} = \underbrace{\langle A(v), w \rangle}_{\text{form on V}}$$

In orthonormal basis, the matrix of A^{dagger} is the complex conjugate transpose of the matrix of A.

In the case A : V —> V, we say that V is:

* Hermitian if A = A^{dagger}

* unitary if A^{-1} = A^{dagger}

* normal, if A commutes with A^{dagger}.

Final exam 10:30 AM - 12:30 PM Thursday April 28.

I'll send an e-mail about policies

Exam will cover the whole course.

Theorem: Normal matrices always hae orthonormal eigenbases.

The eigenvalues are real <—> the matrix is Hermitian.

The eigenvalues have norm 1 <—> the matrix is unitary.