Let V be a real vector space with an inner product $\langle\ ,\ \rangle$. Let $T : V \to V$ be a linear transformation. We say that T is ***self-adjoint*** if $\langle T(\vec{x}), \vec{y}\rangle = \langle \vec{x}, T(\vec{y})\rangle$ for all vectors $\vec{x}$ and $\vec{y}$.

Problem 1. Let V be finite dimensional with orthonormal basis $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_n$. Let A be the matrix of T in this basis. Show that T is self-adjoint if and only if $A_{ij} = A_{ji}$ for all $1 \leq i, j \leq n$.

Let's see what happens if we combine the notions of self-adjointness and eigenvectors:

Problem 2. Let T be self adjoint, and let $\vec{v}$ be a nonzero eigenvector of T. Show that T maps $\vec{v}^{\perp}$ to itself.

Problem 3. Let T be self adjoint, and let $\vec{u}$ and $\vec{v}$ be nonzero eigenvectors of T with distinct eigenvalues α and β. Show that $\langle \vec{u}, \vec{v}\rangle = 0$.

Problem 4. Suppose that T has an orthonormal eigenbasis. Show that T is self-adoint.