Dual space and transpose

Review of dual spaces

Wake up problem: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a linear map, and let $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ be the standard basis of $\mathbb{R}^3$. Suppose that $f(e_1) = 2$, $f(e_2) = 3$ and $f(e_3) = 5$. What is

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In general, a linear map $f : F^n \to F$ is determined uniquely by its value on the basis vectors e_1, e_2, $\ldots$, e_n.

Review of dual spaces

If V is a vector space, then the dual space V^* is $\mathrm{Hom}(V, F)$. If e_1, e_2, ... is a basis of V, then $e_i^* : V \to F$ is the linear function where $e_i^*(\vec{v})$ is the coefficient of e_i in $\vec{v}$.

If $\dim V$ is finite, then the e_i^* are a basis of V^*.

So, for our map $f\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = 2a + 3b + 5c$ on the previous slide, we have $f = 2e_1^* + 3e_2^* + 5e_3^*$. It is often helpful to think of ordinary vectors as column vectors, and dual vectors as row vectors.

So, if $\dim V = n < \infty$, then $\dim V^* = n$ as well.

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More generally,

$$F(f_1^*) = e_1^* + 2e_2^* \qquad F(f_2^*) = 3e_1^* + 4e_2^* \qquad F(f_3^*) = 5e_1^* + 6e_2^*.$$

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So the matrix of F is the transpose matrix, $\left[\begin{smallmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{smallmatrix}\right]$.

In general, suppose we have bases e_1, e_2, ..., e_m of V and f_1, f_2, ..., f_n of W. Let e_i^* and f_j^* be the dual bases of V^* and W^*. Then, if A is the matrix of $F : V \to W$, then A^T is the matrix of $F^* : W^* \to V^*$.

Time for you to talk!

Problem 1 If $F : U \to V$ and $G : V \to W$ are linear maps, then $(GF)^* = F^* G^*$.

Problem 2 If $F : V \to W$ is surjective, then $F^* : W^* \to V^*$ is injective.

Problem 3 If V and W are finite dimensional, and $F : V \to W$ is injective, then $F^* : W^* \to V^*$ is surjective. (This one is harder.)