More on transposes, orthogonal complement

Theorem The rank of a matrix equals the rank of its transpose. Recall that $\mathrm{rank}(A) = \dim \mathrm{Image}(A)$.

We proved this before using row and column reduction; let's give a new proof without coordinates.

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So $\operatorname{rank}(F) = \dim \operatorname{Image}(F) = \dim \operatorname{Image}(F)^* = \dim \operatorname{Image}(F^*) = \operatorname{rank}(F^*)$.

Orthogonal complement

We can use the dual to build something like orthogonal complement without working over the field $\mathbb{R}$. Let V be a vector space and let W be a subspace of V. Then we set $W^\perp$ to be the subspace of V^* defined by

$$W^\perp = \{v^* \in V^* : v^* \text{ is } 0 \text{ on } W\}.$$

In other words, $W^\perp = \mathrm{Ker}(V^* \to W^*)$. (Your book uses W°.)

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Proof: Since $W \to V$ is injective, the map $V^* \to W^*$ is surjective. By rank-nullity, $\dim \operatorname{Ker}(V^* \to W^*) = \dim V^* - \dim \operatorname{Image}(V^* \to W^*) = \dim V^* - \dim W^* = \dim V - \dim W$. $\square$

Time for you to talk!

Problem 1 Let $X \subseteq Y \subseteq V$. Show that $X^\perp \supseteq Y^\perp$ (these are both subspaces of V^*).

Problem 2 Let $W \subset V$ be vector spaces. Show that $(W^\perp)^\perp \supseteq W$.

This one is a bit broken: $(W^\perp)^\perp$ is in V^{**}, not in V. So this only makes sense if we identify V and V^{**}, which only works in finite dimensions, or if we ask that the natural map $V \to V^{**}$ carries W into $(W^\perp)^\perp$, which is true.

Problem 3 Let $W \subset V$ be finite dimensional vector spaces. Show that $(W^\perp)^\perp = W$.

Problem 4 Let X and Y be subspaces of V. Show that $(X + Y)^\perp = X^\perp \cap Y^\perp$. If V is finite dimensional, show also that $(X \cap Y)^\perp = X^\perp + Y^\perp$.

Problem 5 Let V and W be vector spaces and let $A : V \to W$ be a linear transformation. Then $\mathrm{Ker}(A^*) = \mathrm{Im}(A)^\perp$. If V and W are finite dimensional, we also have $\mathrm{Im}(A^*) = \mathrm{Ker}(A)^\perp$.