Bilinear and multilinear forms

Let V and W be vector spaces over a field F. Let B be a function whose input is a vector from V and a vector from W, and whose output is in F.

B is called ***bilinear*** if

$$B(\vec{v}_1 + \vec{v}_2, \vec{w}) = B(\vec{v}_1, \vec{w}) + B(\vec{v}_2, \vec{w})$$

$$B(\vec{v}, \vec{w}_1 + \vec{w}_2) = B(\vec{v}, \vec{w}_1) + B(\vec{v}, \vec{w}_2)$$

$$B(c\vec{v}, \vec{w}) = B(\vec{v}, c\vec{w}) = cB(\vec{v}, \vec{w}) \text{ for } c \in F$$

Wake up question: Let $V = W = \mathbb{R}^2$. Let B be a bilinear function with

$$B\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right) = 3,\ B\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) = 4,\ B\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right) = 5,\ B\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) = 6.$$

What is

$$B\left(\left[\begin{smallmatrix}x_1\\y_1\end{smallmatrix}\right],\ \left[\begin{smallmatrix}x_2\\y_2\end{smallmatrix}\right]\right) \qquad ?$$

$$B\left(\left[\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right],\ \left[\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right]\right) =$$

$$B(x_1\vec{e}_1 + y_1\vec{e}_2,\ x_2\vec{e}_1 + y_2\vec{e}_2) =$$

$$x_1 B(e_1,\ x_2\vec{e}_1 + y_2\vec{e}_2) + y_1 B(e_2,\ x_2\vec{e}_1 + y_2\vec{e}_2) =$$

$$x_1 x_2 B(e_1, e_2) + x_1 y_2 B(e_1, e_2) + x_2 y_1 B(e_2, e_1) + y_2 y_1 B(e_2, e_2) =$$

$$3x_1x_2 + 4x_1y_2 + 5x_2y_1 + 6y_1y_2.$$

In general, if $B : V \times W \to F$ is a bilinear form, $\vec{v}_i$ is a basis of V and $\vec{w}_i$ is a basis of W, then B is determined by the values $B(\vec{v}_i, \vec{w}_i)$. Specifically,

$$B\left(\sum_i a_i \vec{v}_i,\ \sum_j b_j \vec{w}_j\right) = \sum_i a_i B\left(\vec{v}_i, \sum_j b_j \vec{w}_j\right) = \sum_{i,j} a_i b_j B(\vec{v}_i, \vec{w}_j).$$

In general, if $B : V \times W \to F$ is a bilinear form, $\vec{v}_i$ is a basis of V and $\vec{w}_i$ is a basis of W, then B is determined by the values $B(\vec{v}_i, \vec{w}_i)$. Specifically,

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So, if $\dim V = m$ and $\dim W = n$, the space of bilinear pairings $B : V \times W \to F$ forms a vector space of dimension mn.

Thinking in matrices, a bilinear pairing is given by an $m \times n$ matrix Q; we have

$$B(\vec{v}, \vec{w}) = \vec{v}^T Q \vec{w}.$$

A bilinear form is called alternating if $B(\vec{x}, \vec{x}) = 0$ for all vectors $\vec{x}$.

Problem 1: Let B be a bilinear pairing. If B is alternating, then $B(\vec{v}, \vec{w}) = -B(\vec{w}, \vec{v})$ for all vectors $\vec{v}$, $\vec{w}$.

Problem 2: Let B be a bilinear pairing such that $B(\vec{v}, \vec{w}) = -B(\vec{w}, \vec{v})$ for all vectors $\vec{v}$, $\vec{w}$. Suppose that $1 \neq -1$ in the field F. Show that B is alternating.

Let B be an alternating bilinear form on $\mathbb{R}^3$. What relations do we have between:

$$\begin{matrix} B(e_1, e_1) & B(e_1, e_2) & B(e_1, e_3) \\ B(e_2, e_1) & B(e_2, e_2) & B(e_2, e_3) \\ B(e_3, e_1) & B(e_3, e_2) & B(e_3, e_3) \end{matrix} \qquad ?$$

Let B be an alternating bilinear form on $\mathbb{R}^3$.

$$\begin{array}{lll} B(e_1,e_1)=0 & B(e_1,e_2)=-B(e_2,e_1) & B(e_1,e_3)=-B(e_3,e_1) \\ & B(e_2,e_2)=\quad 0 & B(e_2,e_3)=-B(e_3,e_2) \\ & & B(e_3,e_3)=\quad 0 \end{array}$$

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So the space of alternating bilinear forms on $\mathbb{R}^3$ forms a 3-dimensional vector space.

What is the dimension of the space of alternating bilinear forms on $\mathbb{R}^n$?

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So the space of alternating bilinear forms on $\mathbb{R}^3$ forms a 3-dimensional vector space.

What is the dimension of the space of alternating bilinear forms on $\mathbb{R}^n$?

An alternating form is determined by the values $B(e_i, e_j)$ for $1 \leq i < j \leq n$. There are $\binom{n}{2}$ such pairs, so the dimension is $\binom{n}{2}$.

There is much more to say about bilinear forms, both alternating and otherwise. We'll say some of it later in the course. But, for now, we want to move to multilinear forms.

Let V be a vector space. A multilinear form

$$A : \overbrace{V \times V \times \cdots \times V}^{k} \to F$$

is a function which takes in k input vectors and gives a scalar such that, in each position, we have

$$A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{x} + \vec{y}, \vec{v}_{j+1}, \ldots, \vec{v}_k) =$$
$$A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{x}, \vec{v}_{j+1}, \ldots, \vec{v}_k) + A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{y}, \vec{v}_{j+1}, \ldots, \vec{v}_k).$$

$$A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, c\vec{z}, \vec{v}_{j+1}, \ldots, \vec{v}_k) = cA(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{z}, \vec{v}_{j+1}, \ldots, \vec{v}_k).$$

This is just the same linearity condition we had before, in every position.

A multilinear form is alternating if, whenever two of the vectors $\vec{v}_i$ and $\vec{v}_j$ are equal, we have $B(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = 0$.

As before, this implies that

$$B(\vec{v}_1, \cdots, \vec{v}_i, \cdots, \vec{v}_j, \cdots, \vec{v}_k) = -B(\vec{v}_1, \cdots, \vec{v}_j, \cdots, \vec{v}_i, \cdots, \vec{v}_k).$$

Problem: Suppose that $A : V \times V \times V$ is alternating and $A(\vec{x}, \vec{y}, \vec{z}) = 7$. Compute

$$A(\vec{x}, \vec{y}, \vec{z}),\ A(\vec{x}, \vec{z}, \vec{y}),\ A(\vec{y}, \vec{x}, \vec{z}),\ A(\vec{y}, \vec{z}, \vec{x}),\ A(\vec{z}, \vec{x}, \vec{y}),\ A(\vec{z}, \vec{y}, \vec{x}).$$

Problem: Suppose that $A : V \times V \times V$ is alternating and $A(\vec{x}, \vec{y}, \vec{z}) = 7$. Compute

$$A(\vec{x}, \vec{x} + 2\vec{y}, \vec{x} + 2\vec{y} + 3\vec{z}) \text{ and } A(\vec{x}, 2\vec{x} + \vec{y}, 3\vec{x} + 2\vec{y} + \vec{z}).$$

In general, if e_1, e_2, ..., e_n is a basis for V, then an alternating multilinear form on k-vectors is determined by its value on $(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. It is not immediately obvious, though, that, given specified values on $(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$, there is a way to define such an alternating form.

We'll prove that in a moment .... Once we do, we'll know that the space of alternating k-fold multilinear forms from V^k to F is $\binom{n}{k}$.

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Corollary: If $k > \dim V$, the only alternating multinear form $V^k \to F$ is 0.

Corollary: If $k = \dim V$, there a one dimensional space of alternating multinear form $V^k \to F$.

Okay, clearing up the last detail. Suppose we know that $A(e_1, e_2, e_3, e_4, e_5) = 7$. What should $A(e_3, e_5, e_1, e_2, e_4)$ be?

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$$\begin{aligned} A(e_3, e_5, e_1, e_2, e_4) &= -A(e_3, e_4, e_1, e_2, e_5) \\ &= A(e_3, e_2, e_1, e_4, e_5) \\ &= -A(e_1, e_2, e_3, e_4, e_5) = -7 \end{aligned} .$$

What if we did the switching in a different order? Would we get the same sign? It turns out that, yes, we would!

What we need to show is that there is a way to assign a sign to every permutation of $\{1, 2, 3, \ldots, k\}$ such that, switching the order of any two elements, switches the sign. For example:

$$\begin{array}{rrrr} (1,2,3) \rightsquigarrow & 1 & (1,3,2) \rightsquigarrow & -1 \\ (2,1,3) \rightsquigarrow & -1 & (2,3,1) \rightsquigarrow & 1 \\ (3,1,2) \rightsquigarrow & 1 & (3,2,1) \rightsquigarrow & -1 \end{array}$$

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Here is the rule: The sign of $(\sigma(1), \sigma(2), \ldots, \sigma(k))$ is

$$(-1)^{\#\{(i,j)\ :\ i<j \text{ and } \sigma(i)>\sigma(j)\}}.$$

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We need to show that, if we switch $\sigma(i)$, $\sigma(j)$, then this changes sign.

Of course, we switch the contribution from (i, j). Also, if $i < h < j$ and $\sigma(h)$ is between $\sigma(i)$ and $\sigma(j)$, the contribution to the exponent from (i, h) and (h, j) will switch by 2, so the sign will not change.

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Now we know that there is a well defined sign for each permutation. So, given the values $A(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we can describe A on all k-tuples of basis vectors as follows:

If two of j_1, j_2, …, j_k are equal, then $A(e_{j_1}, e_{j_2}, \ldots, e_{j_k}) = 0$.

If all of the j_1, j_2, …, j_k are distinct, and $j_{\sigma(1)} < j_{\sigma(2)} < \cdots < j_{\sigma(k)}$, then

$$A(e_{j_1}, e_{j_2}, \ldots, e_{j_k}) = \mathtt{sign}(\sigma) A(e_{j_{\sigma(1)}}, e_{j_{\sigma(2)}}, \ldots, e_{j_{\sigma(k)}}).$$

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So any specified $\binom{n}{k}$ values for $A(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ do extend to a multilinear form, and the vector space of multilinear forms is $\binom{n}{k}$ dimensional, as desired.